Aspects of F-Theory and M-Theory

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Abstract

Non-perturbative phenomena have received much attention in string theory in the last years. M-Theory and F-Theory are the two main frameworks in which it is possible to explore such phenomena. This thesis focuses on aspects of both theories.

In the first part of this thesis we study F-Theory compactifications with additional abelian gauge symmetries. This was motivated by problems affecting usual F-Theory compactifications and 4-dimensional Grand Unified Theories such as the presence of proton decay operators, which could in principle be resolved with additional abelian symmetries. In the F-Theory context, this translated into the novel analysis of elliptic fibrations with additional (two, in particular) rational sections. A systematic study of the possible degenerations of such elliptic fibrations through the application of Tate's algorithm was carried out and provided new insight into the phenomenology of F-Theory models with additional U(1) factors.

The second part of this thesis consists of the study of some aspects of membranes in M-Theory. D-branes in string theory are well understood thanks to a perturbative definition via open strings. On the contrary, membranes and fivebranes in M-Theory lack such a description and their effective theories are not as well understood.

In particular the theory on parallel M5-branes, the so-called (2,0) theory, was studied in some detail. Following a number of results and dualities in lower dimensional field theories obtained in the last years starting from the (2,0) theory, the latter was compactified on a 2-dimensional sphere to obtain a 4-dimensional sigma model into the moduli space of monopoles. A supergravity background was turned on in order to preserve supersymmetry and an intermediate reduction to 5-dimensional $\mathcal{N} = 2$ Super-Yang-Mills theory was used by considering the two-sphere as a circle fibration over an interval.

Insight into the theory on parallel M5-branes was also gained by relating it to the better known dynamics on coincident M2-branes. This followed a recent proposal for the realization of the (2,0) algebra on a non-abelian tensor multiplet through the use of 3-algebras. In this thesis we generalize this proposal and find an algebraic structure which describes two parallel M5-branes or two parallel M2-branes depending on whether a particular abelian three-form is turned on.

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Chapter 1

Introduction

String theory has been the main attempt in the quantum gravity program to try to provide a unified description that would include both gravity and the known gauge interactions described by the Standard Model. One of the appeals to string theory was supposed to be that, through M-Theory, it was to be uniquely defined and therefore expected to provide a single consistent description of the physics beyond the Standard Model. As it turned out, this is not the case, since even if the theory in 11 dimensions is unique, the theories resulting from compactifications down to 4 dimensions are incredibly numerous, thus creating an important problem in the extension of our physical knowledge beyond the Standard Model. On the opposite end of trying to correctly reduce the higher dimensional theory to 4 dimensions to reproduce the Standard Model, lies a yet not complete understanding of M-Theory in its own right. Indeed, the fact that M-Theory only allows a non-perturbative regime created an obstacle into obtaining insight into the full dynamics of the theory. Non-perturbative phenomena have therefore been a very important aspect of research in the string theory program in the last years. This thesis tries to give a contribution to the two main lines of research just detailed, that is, the phenomenological reduction to 4 dimensions and the better understanding of non-perturbative phenomena of string theory.

Before 1994, five 10-dimensional string theories were known, obtained by quantizing the superstring and applying different projections for the states. The insight provided by Witten ([2]) was then to understand the strong coupling regime of Type IIA string theory as an 11-dimensional theory, then called M-Theory, whose circle reduction would reproduce the perturbative regime of Type IIA. In particular the radius of the circle R

was related to the string coupling of Type IIA via

$$R = g_s l_s, \tag{1.0.1}$$

where l_s is the string length. The existence of an 11-dimensional supergravity theory which could serve as the low energy theory of M-Theory seemed to confirm such proposal, thanks also to the fact that the circle reduction of 11-dimensional supergravity correctly reproduces Type IIA supergravity. Nevertheless, the absence of a coupling constant presented a major difficulty for the understanding the dynamics of the theory itself. Such fact was actually signaling the absence of strings themselves as fundamental objects, and it was soon understood that they were to be replaced by membranes and fivebranes, the respectively two and five (spatial) dimensional BPS solutions of 11-dimensional supergravity.

Even though the low energy theories describing parallel D-branes can now be understood in detail, and can actually all be derived from 10-dimensional Super-Yang-Mills theory, a simple generalization did not appear manifest for the case of membranes and fivebranes. This was of course due to the absence of open strings themselves, which allowed in the case of D-branes a perturbative definition via string scatterings. This left an important theoretical gap in the understanding of the full structure of M-Theory and how it encodes the full set of non-perturbative phenomena of 10-dimensional string theories. Much work has been dedicated to gaining more insight into the description of parallel branes in M-Theory, and significant progress has started to be achieved in the last few years.

The first breakthrough was realized by the BLG model ([3, 4]) which correctly reproduced the dynamics of two coincident membranes, or M2-branes. Such a theory was supposed to satisfy a number of requirements, such as preserving $\mathcal{N} = 8$ supersymmetry (M2-branes being half BPS objects of 11-dimensional supergravity), being a conformal theory (since there is no characteristic length in M-Theory), correctly reproducing the particular scaling of the entropy with the number N of parallel membranes (which was known to be proportional to $N^{3/2}$) and still allowing a non-trivial interaction between the degrees of freedom of the theory. The BLG model successfully satisfied all such requirements. Surprisingly, it did so through the introduction of a novel gauge symmetry, which relies on 3-algebras rather than conventional Lie algebras. Such algebraic structures are characterised by a totally antisymmetric triple bracket which acts as a derivation on a vector space, thus generalizing the conventional Lie bracket. Successively, a correct description was found for an arbitrary number of parallel M2-branes through the ABJM model ([5]), albeit in an orbifold background $\mathbb{C}^4/\mathbb{Z}_k$.

The low energy theory describing parallel fivebranes, or M5-branes, has instead presented more difficulties and a satisfactory description is still lacking. Nevertheless, the (2,0) theory (the theory on parallel M5-branes), has produced a number of results in lower dimensional field theories which are independent of the precise formulation of the theory. In particular, different compactifications of the (2,0) theory have given rise to surprising dualities between theories in different dimensions, therefore providing insight into such field theories themselves. For what concerns a formulation of the non-abelian (2,0) theory itself, progress has been made recently through the realization of a set of equations of motion for a non-abelian tensor multiplet which is invariant under (2,0) supersymmetry in 6 dimensions ([6]). As in the case of the BLG model, the gauge symmetry is based on a 3-algebra rather than usual Lie algebras and such proposal aims to correctly describe the dynamics of two M5-branes. Among the difficulties in providing a Lagrangian description lies nevertheless the presence of a self-dual three-form field strength, and it is believed that such description is not actually possible.

Therefore, in the context of M-Theory, one of the main directions of research has been to gain a full understanding of the dynamics of coincident branes and to shed light on non-perturbative phenomena arising in string theory.

F-Theory ([7–9]) is a second framework in which non-perturbative phenomena can be taken into account and which has served a great purpose for the geometric engineering of 4-dimensional theories obtained by compactifications. In 1996 Vafa ([7]) interpreted for the first time the $SL(2,\mathbb{Z})$ invariance of Type IIB string theory as the modular group of an auxiliary torus assigned to every point of the internal space-time. In particular, the axiodilaton field τ of Type IIB string theory was interpreted as the complex structure of such torus, and compactifications in the presence of 7-branes were studied. At the locus where the 7-branes are located, the axiodilaton is found to diverge and it therefore followed that the torus described by such complex structure is not well defined, and is actually singular. The picture which arises this way is that of a fibration of space-time by complex tori, an elliptic fibration, which becomes singular at the location of the 7-branes. This is not actually describing a physical theory in 12 dimensions, for which there would be no low energy supergravity approximation, but rather a geometric framework for taking into account the (non) perturbative effects arising in compactifications of type IIB string theory in the presence of 7-branes. Note that F-Theory also allows definitions through dualities with M-Theory or with $E_8 \times E_8$ Heterotic string theory.

Therefore the study of the properties of the resulting compactification is translated into the study of the geometric properties of the elliptic fibration, which represents the internal space of the compactification and the two fictitious dimensions of the elliptic fiber. In particular, the gauge group, the matter content and the Yukawa couplings of the 4-dimensional $\mathcal{N} = 1$ theory, which results from compactifying F-Theory on a Calabi Yau four-fold, are nicely encoded in the singularity structure of the elliptic fibration in codimension one, two and three respectively. From a phenomenological perspective, F-Theory allows to geometrically engineer (that is, to model a theory based on the geometric properties of the compactification manifold) a whole class of 4-dimensional supersymmetric theories. Such a contribution fits into the study of Grand Unified Theories (GUTs), a program which, independently from string theory, had tried to unify the known gauge interactions of the Standard Model into a single gauge group of a supersymmtric theory. Indeed, contrary to what happens in the Standard Model, in $\mathcal{N} = 1$ supersymmetric theories in 4 dimensions, such as the Minimal Supersymmetric Standard Model, the running of the coupling constants under the RG flow results in the intersection in a single point at an energy around 10^{16} GeV. This can be interpreted as the existence of a single gauge group at higher energies which then breaks at 10^{16} GeV to the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$. Therefore supersymmetric theories which could embed the Standard Model gauge group as a maximal subgroup of a single gauge group started to be proposed as models for the unifications of the known gauge interactions and are known as GUTs.

Even though supersymmetric theories could solve a number of problems afflicting the Standard Model and could also provide a surprising way in which the known gauge interactions could be united, they were also afflicted by their own problems. It was realized that unwanted operators could result by the embedding of the Standard Model gauge group in a single group, which could not be reconciled in any way with empirical observations. The main such case is represented by the proton decay operator which arises in Grand Unified Theories and which predicts a non-zero half life for the proton. This is in stark contrast with experiments which have ruled out such eventuality by asserting that the half life of the proton cannot be smaller than the age of the universe.

Surprisingly F-Theory provides a way to obviate such a problem, again through geometric properties of the compactification manifolds. Indeed, it can be shown that if the elliptic fibrations admits extra rational sections (that is, extra divisors which are copies of the base of the fibration), additional abelian gauge factors are introduced in the resulting theory in 4 dimensions. Such abelian factors are fundamental in getting rid of proton decay operators, as they can prevent them from being gauge invariant and therefore not physically realized. Therefore F-Theory can be shown to be a successful framework in which unwanted phenomena afflicting GUTs can be taken into account.

In this thesis the lines of research just detailed are expanded in more detail and tentative contributions to resolving such questions are presented as follows. In Chapter 2 an extended summary of F-Theory notions is presented, to serve as an introduction to the results of ([10]). Chapter 3 is then largely based on the work carried out in ([10]), where the singularity structure of a class of elliptic fibration with two additional rational sections is studied through the so-called Tate's algorithm. In the second part of this thesis, the focus is switched to the study of some aspects of M-Theory. In Chapter 4 we review known facts about M-Theory and its fundamental objects, membranes and fivebranes. Chapter 5 presents the original results of ([11]), where a novel representation of the (2,0) algebra in 6 dimensions was realized on a non-abelian tensor multiplet and was found to be related to the BLG model describing two M2-branes by a natural dimensional reduction. Finally, Chapter 6 presents some results arising from an early collaboration toward the work realized in ([12]) and studies the reduction of the (2,0) theory describing parallel M5-branes on a two-sphere, resulting in a sigma model into the moduli space of centered SU(2) monopoles.

Chapter 2

Aspects of F-Theory

F-Theory is a geometric framework which takes into account the backreaction of 7-branes on space-time in type IIB string theory. This will be our starting point in trying to understand how F-Theory takes into account non-perturbative effects which need to be considered in type IIB compactifications. Indeed, in ordinary compactifications of type II string theories in the presence of branes, the backreaction of the latter on spacetime is usually neglected. This is legitimate as long as the codimension of the brane is different from two. One of the main reasons why F-Theory is necessary as a framework for studying configurations of 7-branes can be traced to the different dependence of solutions to the sourcing Poisson equation for 10-dimensional fields. In the presence of a brane the fields are sourced by the backreaction of the brane on spacetime. In particular we have

$$\begin{aligned} &\Delta\Phi(r) \simeq \delta(r) \longrightarrow \Phi(r) \simeq \frac{1}{r^{7-p}}, \quad \text{p-branes} \\ &\Delta\Phi(r) \simeq \delta(r) \longrightarrow \Phi(r) \simeq \log(r), \quad \text{7-branes}, \end{aligned} \tag{2.0.1}$$

where Φ is a generic space-time field and r is the distance from the brane. We see that branes are sources for space-time fields, and solutions to the corresponding Poisson equations scale accordingly to the type of brane we are looking at. Such a backreaction of the branes on the space-time fields can be neglected as long as we are not considering 7-branes. In that case, the approximation is not valid since the fields scale as $\Phi(r) \simeq \log(r)$, which does not become negligible as we move away from the brane.

Therefore we see that we have a fundamental problem in considering type IIB compactifications in the presence of 7-branes, in as much as the perturbative regime is not valid and we do not know how to gain full insight into the 4-dimensional theories arising from such compactifications, which are of phenomenological interest. We gave a heuristic explanation as to why a framework for taking into account non-perturbative effects of type IIB string theory is necessary.

In this chapter we provide a background understanding of F-Theory, including a presentation of the mathematics behind it. We review concepts in the geometry of elliptic fibrations and their singularities, in order to understand how F-Theory takes into account non-perturbative effects of Type IIB string theory. Finally, we address the problem of developing additional abelian factors in Grand Unified Theories through F-Theoretic methods. This will turn out to be of relevance for phenomenological reasons.

2.1 $SL(2,\mathbb{Z})$ and Type IIB String Theory

Recall the type IIB field content. We have the metric $g_{\mu\nu}$, the Kalb-Ramond 2-form $B_{\mu\nu}$, the dilaton ϕ and potentials C_p for p even, denoting coupling to odd dimensional branes. If we define the axiodilaton as

$$\tau \equiv C_0 + ie^{-\phi}, \tag{2.1.1}$$

the action can be written in the Einstein frame, where $G_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$, as ([13])

$$\begin{aligned} S_{IIB} = & \frac{1}{2k^2} \int d^{10}x\sqrt{-g}\left(R - \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2\,\mathrm{Im}(\tau)^2} - \frac{|G_3|^2}{2\,\mathrm{Im}(\tau)} - \frac{|\tilde{F}_5|^2}{4}\right) \\ & + \frac{1}{8ik^2}\int \frac{C_4 \wedge G_3 \wedge \tilde{G}_3}{\mathrm{Im}(\tau)} \end{aligned} \tag{2.1.2}$$

where

$$H_3 = dB_{\mu\nu} \qquad F_{p+1} = dC_p, \tag{2.1.3}$$

and the following combinations were also defined

$$\begin{aligned} \tilde{F}_3 &= F_3 - C_0 \wedge H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} F_2 \wedge F_3 \\ G_3 &= F_3 - \tau H_3. \end{aligned} \tag{2.1.4}$$

We can then define $SL(2,\mathbb{Z})$ transformations represented by matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(M) = 1, \quad \{a,b,c,d\} \in \mathbb{Z}. \tag{2.1.5}$$

The action on the axiodilaton is

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{2.1.6}$$

while the doublet ($C_2 \equiv C_{\mu\nu}, B_{\mu\nu}$) transforms as

$$\begin{pmatrix} C_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix}. \tag{2.1.7}$$

The other fields are invariant under such transformations. The action (2.1.2) is then invariant under the group $SL(2,\mathbb{Z})$, and is actually invariant under the larger group $SL(2,\mathbb{R})$, but such a symmetry breaks at the quantum level to $SL(2,\mathbb{Z})$. Let us look at the generators of the group $SL(2,\mathbb{Z})$. They are

$$SL(2,\mathbb{Z}) = \left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle. \tag{2.1.8}$$

It will be relevant to look at how such generators act on the axiodilaton, which can be re-written in terms of the string coupling as

$$g_s = e^{\phi}, \quad \tau = C_0 + \frac{i}{g_s}. \tag{2.1.9}$$

T transformations do not affect the string couplings and only operate a shift in C_0. On the other hand, we see that under S transformations the axiodilaton transforms as

$$\tau \to -\frac{1}{\tau}. \tag{2.1.10}$$

The effect of such a transformation on the string coupling can be analysed in a simple background with $C_0 = 0$ to see that

$$g_s \to \frac{1}{g_s}, \tag{2.1.11}$$

therefore giving rise to a weak-strong duality. We will now see how these dualities come into play in the presence of 7-branes.

Consider a compactification set up in type IIB string theory where we split space-time into $\mathbb{R}^{1,3} \times M_6$, where M_6 is the internal manifold. Moreover, let a 7-brane wrap $\mathbb{R}^{1,3} \times M_4$, with M_4 a four-cycle inside M_6, that is

$$\begin{cases} \text{Type IIB} & : \; \mathbb{R}^{1,3} \times M_6 \\ \text{7-brane} & : \; \mathbb{R}^{1,3} \times M_4 \end{cases}, \qquad M_4 \subset M_6.$$

Let the complex coordinate z parametrize the transverse direction to the 7-brane in the ambient space-time, and let the 7-brane be located at z_0. As explained, the brane is a source for the space-time fields, and in particular ([14]), C_8 receives a correction through the following Poisson equation

$$d \star F_9 \simeq \delta^2(z - z_0), \quad F_9 = dC_8. \tag{2.1.12}$$

Let us integrate this equation over the whole complex plane to find

$$\int_{\mathbb{C}} dF_1 = 1 \qquad F_1 \equiv \star F_9. \tag{2.1.13}$$

We can then apply Stokes theorem to turn the left-hand side into a contour integral about the position of the 7-brane

$$\oint_{S^1} dC_0 = 1. \tag{2.1.14}$$

We can find a solution to this equation given by

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} \log(z - z_0) + \ldots, \tag{2.1.15}$$

where the ellipses denotes terms regular in z which do not contribute to the contour integral. But this raises a problem, as in encircling the 7-brane in the transverse direction, we see that τ changes as

$$\tau \to \tau + 1. \tag{2.1.16}$$

The presence of such a monodromy would turn τ into a multivalued function, therefore making it quite hard to interpet τ as a space-time field. But as we have seen already, the situation is saved by the $SL(2,\mathbb{Z})$ invariance of type IIB, so that the value of the axiodilaton after encircling a 7-brane is the same up to a $SL(2,\mathbb{Z})$ transformation.

We have seen how the $SL(2,\mathbb{Z})$ invariance of type IIB string theory plays an important role in the presence of 7-branes, guaranteeing that we can make sense of the monodromy arising from the backreaction of the brane on space-time. We will now see how the $SL(2,\mathbb{Z})$ group arises in the description of complex tori.

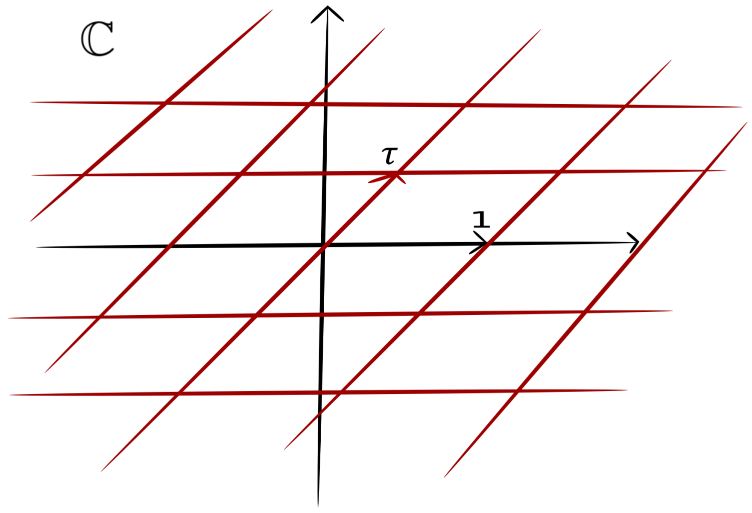


Figure 2.1: The fundamental domain associated to the torus with complex structure τ.

2.2 $SL(2,\mathbb{Z})$ and F-Theory

The group $SL(2,\mathbb{Z})$ is better known in the mathematics literature for its relation to complex tori in the definition of the modular parameter τ. We can always find for a differentiable torus, that is a Riemann surface of genus one, a complex structure. To get a more concrete insight into this statement, we can view a torus as the following quotient of the complex plane

$$\mathbb{T}^2 = \mathbb{C}/\Lambda, \tag{2.2.1}$$

where Λ is an integer lattice, that is $\Lambda \simeq \mathbb{Z} \oplus \mathbb{Z} = \{a\mathbb{Z} + b\mathbb{Z}\} \subset \mathbb{C}$. We can always rescale such a lattice so that the first defining vector can be taken to be the unit vector and the second defining vector can be taken to be τ, see Figure 2.1.

It is not hard to see that sending

$$T: \quad \tau \to \tau + 1, \tag{2.2.2}$$

leaves the lattice unchanged. In a similar fashion, it can be shown that the transformations

$$S: \quad \tau \to -\frac{1}{\tau}, \tag{2.2.3}$$

just flip the sides of the lattice and therefore do not affect the lattice describing the torus. The modular group is the group generated by such transformations, which are seen to obey

$$S^2 = \mathbf{1}, \quad (ST)^3 = \mathbf{1}. \tag{2.2.4}$$

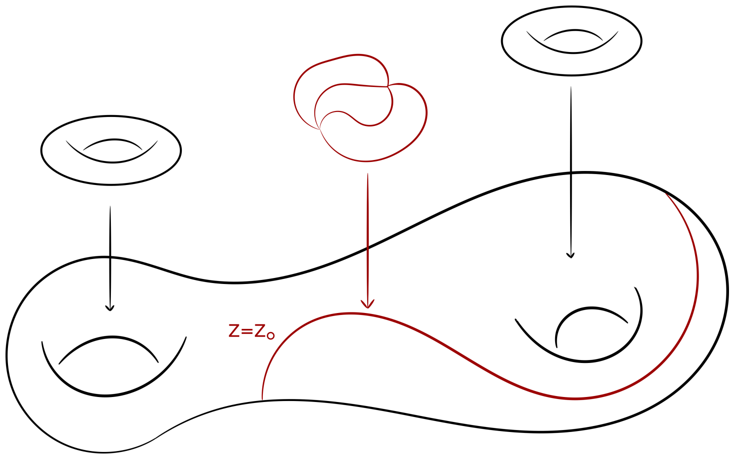


Figure 2.2: The elliptic fibration becomes singular at the locus $z = z_0$, where the 7-brane is located.

and which can be represented by matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(M) = 1, \quad \{a, b, c, d\} \in \mathbb{Z}. \tag{2.2.5}$$

Therefore we have a one-to-one correspondence between equivalent tori and conjugacy classes of complex structures modulo the action of the modular group. This defines the moduli space of complex tori as $\mathbb{C}/SL(2, \mathbb{Z})$, which is the usual fundamental region of the upper complex plane.

What we are interested in here, though, is the fact that each complex number τ defines the complex structure of a torus up to the action of the $SL(2, \mathbb{Z})$ group. This is exactly the situation that we found in analysing the axiodilaton τ in type IIB string theory in the presence of 7-branes. F-theory will take the hint from this appearance of the modular group as both a symmetry group of type IIB string theory, and as the mapping class group of complex tori, to give a new interpretation of the axiodilaton field.

Recall the situation described so far. We noted that type IIB string theory possesses an $SL(2, \mathbb{Z})$ invariance, and we also noted that the axiodilaton undergoes such transformations in the presence of 7-branes. Or equivalently, 7-branes generates monodromies for τ which can be reabsorbed by an $SL(2, \mathbb{Z})$ transformation. On the other hand we saw that the complex number τ describes one and only one torus up to the action of the $SL(2, \mathbb{Z})$ group, which is a symmetry of type IIB string theory. F-Theory ([7–9]) takes this hint seriously and interprets the axiodilaton as the complex structure of a torus. As the axiodilaton varies over space-time, so does the complex struture of the torus. Effectively, we are associating to each point of space-time a torus, that is, we are fibering space-time with an elliptic fibration. At the location of the 7-brane something particular happens. Recall that (2.1.15)

$$\Delta\tau \simeq \frac{1}{2\pi i}\log(z - z_0). \tag{2.2.6}$$

We see that at the location z_0 of the brane, the complex structure of the torus diverges, or equivalently, the torus becomes singular, see Figure 2.2. In the next section we will go in detail into the mathematics of elliptic fibrations and the possible singularities that may occurr.

Such an elliptic fibration is not to be interpreted as a description of a 12-dimensional theory, as there is no supergravity which could describe its low energy dynamics. It should instead be understood as a bookkeeping device to study type IIB string theory in its different regimes of coupling. Notice that, even though the complex structure diverges at the location of the brane, the string coupling does not vanish there ([14]). The ambiguity is due to the casting of the type IIB action in the Einstein frame, but as usual the coupling of the brane theory is proportional to the volume of the cycle wrapped by the brane.

2.3 Elliptic Curves

As we saw in the previous section, in order to understand 7-branes configurations, F-Theory understands the axiodilaton as the varying complex structure of a torus associated to each point of the internal space. This gives rise to an elliptic fibration, and in this section we are going in some details into the mathematics describing such constructions.

An elliptic fibration is a fibration such that the generic fiber is an elliptic curve (nevertheless we will be interested in the non-generic fiber, that is, in singular fibers). We write this as

$$\begin{array}{rcl} \mathbb{E} \hookrightarrow Y & \qquad & \mathbb{E} = \text{Elliptic curve} \\ \downarrow & & Y = \text{Total Space} \\ B & & B = \text{Base of the Fibration} \end{array} \tag{2.3.1}$$

where $\mathbb{E}$, the fiber, is an elliptic curve, Y is the total space of the fibration, which projects onto the base B. First it will be necessary to spend some time describing elliptic curves, their relations to complex tori and their expressions as subsets of projective spaces. With this background we will then be able to approach elliptic fibrations and study the conditions for which these are well defined, and their possible degenerations.

An elliptic curve is an algebraic curve of genus one with a specified point O which is non-singular and which is projective, that is, which can be described as a subset of a projective space ([15]). Since we will be interested in elliptic curves over the complex numbers, we will also make clear the relation between complex tori, $\mathbb{C}/\Lambda$, and elliptic curves over $\mathbb{C}$, $\mathbb{E}/\mathbb{C}$. An elliptic curve $\mathbb{E}/\mathbb{C}$ will turn out to have a standard form, called the Weierstrass form, in which it is always possible to be cast. To this end recall the definition of complex projective space $\mathbb{P}^n$ as the quotient

$$\mathbb{P}^n = \mathbb{C}^n/\mathbb{C}^*, \tag{2.3.2}$$

where the action of $\mathbb{C}^*$ defines the equivalence relation we mod out by as

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \Leftrightarrow (x_1, \ldots, x_n) = \lambda(y_1, \ldots, y_n) \qquad \lambda \neq 0. \tag{2.3.3}$$

The Weierstrass form allows to cast every elliptic curve into the form

$$\mathbb{E}: \qquad y^2 = x^3 + fx + g, \tag{2.3.4}$$

where we work in the affine patch of $\mathbb{P}^2 = [x : y : z]$ given by $z = 1$.

In the next section we will find two ways to bring an elliptic curve into the Weierstrass form. Through one of these we will also show the equivalence of complex tori and elliptic curves over the complex numbers. Through the second we will introduce algebro-geometric methods that will be useful in the description of elliptic fibrations.

2.4 Weierstrass Form for Elliptic Curves

In order to cast an elliptic curve into Weierstrass form (2.3.4) we will actually show the equivalence between complex tori and elliptic curves over the complex numbers, so that we will effectively get a twofold result. So let us start from a complex torus given by $\mathbb{T}^2 = \mathbb{C}/\Lambda$; we would like to find a function to (an affine patch of) projective space which is well defined and bijective

$$\Phi: \ \mathbb{C}/\Lambda \ \longleftrightarrow \ \mathbb{E}/\mathbb{C}. \tag{2.4.1}$$

Consider the first direction: we need to find a function which is well defined on the lattice Λ, that is, a doubly periodic function. The function we will use goes back to Weierstrass and can be written in the form

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda, w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right). \tag{2.4.2}$$

Through an expansion in Laurent series of $\wp(z)$ and its derivative $\wp'(z)$ it can be proved that the following relation holds

$$\wp'(z)^2 = \wp(z)^3 + f\wp(z) + g, \tag{2.4.3}$$

where we omit the expansion of f and g in terms of Eisenstein series. Therefore we can define the following map

$$\begin{aligned} \Phi: \ \mathbb{C}/\Lambda \longrightarrow & \qquad \mathbb{P}^2 \\ z \longrightarrow & \ [\wp(z), \wp'(z), 1] \end{aligned} \tag{2.4.4}$$

which is bijective and well defined between the complex torus and the codimension one subset of $\mathbb{P}^2[x : y : z]$ defined by the relation

$$y^2 = x^3 + fx + g. \tag{2.4.5}$$

Notice that the map Φ is well defined as long as the right hand side of (2.4.3) has different roots, that is the discriminant of the equation

$$x^3 + fx + g = 0 \tag{2.4.6}$$

is non-vanishing. This turns out to be a very important quantity in its own right

$$\Delta = 4f^3 + 27g^2. \tag{2.4.7}$$

The subset identified by Φ to be in bijective correspondence with a complex torus is what we call an elliptic curve. This is indeed an algebraic projective curve of genus one (since the torus is a Riemann surface of genus one), whose smoothness turns out to be guaranteed by the non-vanishing of the discriminant, and which has a specified point. This is the so-called point at infinity and is given by $[1 : 1 : 0]$.

Note that this is the case since the minimal way to homogenize the Weierstrass form is by understanding it as a subset of the weighted projective space $\mathbb{P}^2[x : y : z]$ with weights $(2, 3, 1)$ (which we write as $\mathbb{P}^{231}$). Recall that weighted projective space $\mathbb{P}^n$ with weights $(w_1, \ldots, w_n)$ is the usual projective space where we modify the $\mathbb{C}^*$ action to get the equivalence relation between two points of $\mathbb{C}^n$ given by

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \Leftrightarrow (x_1, \ldots, x_n) = (\lambda^{w_1} y_1, \ldots, \lambda^{w_n} y_n) \qquad \lambda \neq 0. \tag{2.4.8}$$

Then it is easily seen that the Weierstrass form can be homogenized to $y^2 = x^3 + fxz^4 + gz^6$ and the point at infinity is indeed a point on the elliptic curve. We have found a bijection between a complex torus and what we defined as an elliptic curve over the complex

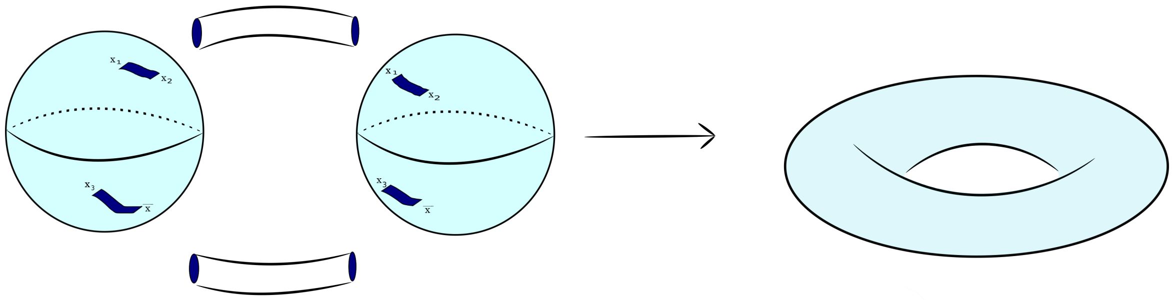


Figure 2.3: A torus as a double sheeted cover of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ branched over four points.

numbers. We saw that in so doing we managed to cast the elliptic curve in the so-called Weierstrass form. This allows another intuition into the equivalence that we showed. Indeed we see that following from the rearranging

$$y = \pm\sqrt{x^3 + fx + g} \tag{2.4.9}$$

we can have a hint of the topology of an elliptic curve over the complex numbers by noting that the double-sheeted cover of (2.4.9) has branch cuts joining (x_1, x_2) and $(x_3, \bar{x} = \infty)$, where $\{x_1, x_2, x_3\}$ are the roots of the right-hand side of (2.4.9). Then we can glue two copies of the Riemann sphere (that is of $\mathbb{P}^1_{\mathbb{C}}$) along the fattened branch cuts, that is we glue two spheres with two disks cut out along the cuts. This gives a torus as in Figure 2.3.

As anticipated we are now going to provide a second way to derive the Weierstrsass form for an elliptic curve using the Riemann-Roch theorem for algebraic curves of genus one. This will turn out to be useful both to introduce algebro-geometric methods which play a role in elliptic fibrations and for generalizations to elliptic curves with multiple points specified that will be studied in Chapter 3.

Let $\mathcal{C}$ be an algebraic curve and let $\mathcal{L}$ be a line bundle over it. The Riemann-Roch theorem relates the dimension of the space of global sections of the line bundle $\mathcal{L}$ to the degree of the line bundle $\mathcal{L}$ and the genus of the algebraic curve $\mathcal{C}$. In particular recall that for a divisor

$$D = \sum_{P \in \mathcal{C}} n_P P \tag{2.4.10}$$

on an algebraic curve $\mathcal{C}$, we define the associated line bundle $\mathcal{O}(D)$ to be the vector space of meromorphic functions with poles at worst of order n_P at P. Then the Riemann-Roch theorem states that

$$\dim \mathcal{O}(D) = \deg(D) + 1 - g, \tag{2.4.11}$$

where g is the genus of the curve $\mathcal{C}$ and

$$\deg(D) = \sum_P n_P. \tag{2.4.12}$$

We see that in particular for an elliptic curve we find

$$\dim \mathcal{O}(D) = \deg(D). \tag{2.4.13}$$

Now let us consider the line bundle $\mathcal{O}(P)$, where P is the specified point on the elliptic curve. As we saw, this is the space of meromorphic functions having at worst a simple pole at P. By Riemann-Roch such a space is 1-dimensional and is spanned by a single section, that we call z. Similarly $\mathcal{O}(2P)$ is seen to be generated by z^2 and a new section which we call x. Following this reasoning we see that

$$\begin{aligned}
&\mathcal{O}(3P) \overset{\text{gen. by}}{\longrightarrow} \{z^3, zx, y\} \\
&\mathcal{O}(4P) \overset{\text{gen. by}}{\longrightarrow} \{z^4, z^2x, zy, y^2\} \\
&\mathcal{O}(5P) \overset{\text{gen. by}}{\longrightarrow} \{z^5, z^3x, z^2y, x^2z, xy\} \\
&\mathcal{O}(6P) \overset{\text{gen. by}}{\longrightarrow} \{z^6, z^4x, z^3y, z^2x^2, y^2, x^3, zxy\},
\end{aligned} \tag{2.4.14}$$

but we immediately see that $\mathcal{O}(6P)$ has naively seven generators, while the Riemann-Roch theorem states that it should be 6-dimensional. Therefore there must be a relation between such generators

$$a_1y^2 + a_2x^3 + a_3z^6 + a_4z^4x + a_5z^3y + a_6z^2x^2 + zxy = 0. \tag{2.4.15}$$

If the characteristic of the field we are working over is different from 2 or 3, we can then complete the square in y and the cube in x to turn the last equation into the Weierstrass form

$$y^2 = x^3 + fxz^4 + gz^6. \tag{2.4.16}$$

We started from a smooth algebraic curve $\mathcal{C}$ of genus one with a specified point, that is an elliptic curve, and applied the Riemann-Roch theorem on the line bundle $\mathcal{O}(P)$ over $\mathcal{C}$. This allowed to find three global sections $\{z, x, y\}$ which can be considered as maps

$$\{z, x, y\} : \mathcal{C} \hookrightarrow \mathbb{P}^2, \tag{2.4.17}$$

that is they provide an embedding of the elliptic curve into projective space. The projective equation describing the elliptic curve is found to be the Weierstrass form (2.4.16).

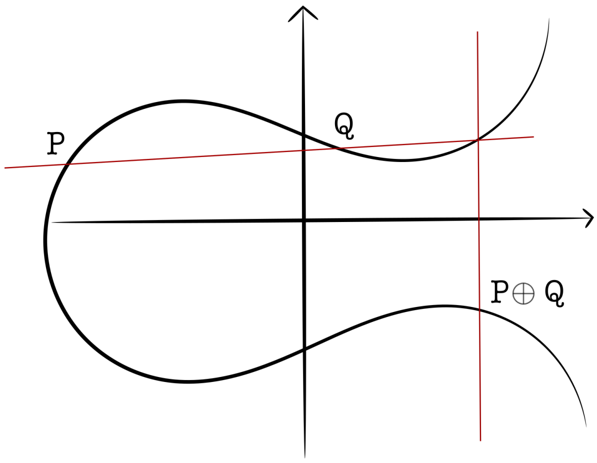


Figure 2.4: A representation of the group law $\oplus$ defined on the set of rational points $\mathbb{E}(K)$ of an elliptic curve. If P and Q are rational points it follows by solving polynomial equations that $P \oplus Q$ is also rational.

2.5 Mordell-Weil Group

When considering elliptic curves over the complex numbers we saw that we could think of them as Riemann surfaces of genus one. Therefore, since we have an obvious group structure on the torus which descends from addition on $\mathbb{C}$ under the quotient by Λ, we might wonder what is the corresponding group structure on the elliptic curve. As it turns out, elliptic curves admit a group structure not only when defined over the complex numbers, but over a generic field K. Let us now discuss such group structure in more detail.

Let $\mathbb{E}(K)$ be the set of K-rational points of the elliptic curve $\mathbb{E}$, that is the set of points in $\mathbb{P}^2_K$ which belong to $\mathbb{E}$. Then the Mordell-Weill theorem states that $\mathbb{E}(K)$ is a group, and in particular it is a finitely generated abelian group ([15]). Every such group is isomorphic to

$$\mathbb{E}(K) \simeq \mathbb{Z}^{\oplus k} \oplus \mathcal{G}_T, \tag{2.5.1}$$

where $\mathcal{G}_T$ is the torsion part (of finite order). We call $\mathbb{E}(K)$ the *Mordell-Weil group* of $\mathbb{E}$, so that k, the dimension of the non-torsion part of $\mathbb{E}(K)$, is the rank of the t Mordell-Weil group of the elliptic curve $\mathbb{E}$. The statement that $\mathbb{E}(K)$ is a group means that there exist a binary operation $\oplus$ on the set $\mathbb{E}(K)$ of rational points of an elliptic curve such that the identity element of the group is the specified point $\mathcal{I}$ on the elliptic curve (which can be taken to be the point at infinity). Then we have that

(*i*) $P \oplus Q = Q \oplus P$ for all $P, Q \in \mathbb{E}(K)$;

(*ii*) $P \oplus \mathcal{I} = P$ for all $P \in \mathbb{E}(K)$;
(*iii*)$(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$ for all $P, Q, R \in \mathbb{E}(K)$;
(*iv*) If $P \in \mathbb{E}(K)$ then there exists $Q \in \mathbb{E}(K)$ such that $P \oplus Q = I$.

Even though from an F-Theory perspective we will be interested mainly in the rank of the Mordell-Weil group, it turns out that there is a graphic description of the group operation $\oplus$ on $\mathbb{E}$. For two generic rational points P and Q in $\mathbb{E}(K)$ we let R be the third intersection of the line between P and Q with the elliptic curve. Then we let $P \oplus Q$ be equal to $-R$, i.e. the intersection of the line between the point at infinity and R, as depicted in Figure 2.4. It is a property of polynomial equations that $P \oplus Q$ is also a rational point of $\mathbb{E}$ (the group law is not well defined if $P \oplus Q$ is taken to be R). Notice that the group structure on the elliptic curve is well defined because we have a specified point to begin with, the identity of the group structure.

2.6 Elliptic Fibrations

Recall that we defined an elliptic fibration as

$$\begin{array}{ccl} \mathbb{E} \hookrightarrow Y & \qquad & \mathbb{E} = \text{Elliptic curve} \\ \downarrow & & Y = \text{Total Space} \\ B & & B = \text{Base of the Fibration} \end{array} \tag{2.6.1}$$

Such a variety has as fiber over each point of the base an elliptic curve described by a Weierstrass form embedded in projective space $\mathbb{P}^{123}$. This is called an E_8 fibration for reasons that will become clear later; there also exist E_7 and E_6 fibrations represented by quartic equations in $\mathbb{P}^{112}$ and cubic equations in $\mathbb{P}^2$ respectively (in order to satisfy the Calabi-Yau condition the homogeneous degree of the equation describing a projective variety should equal the sum of the weights of the ambient projective space). Such fibrations will have Mordell-Weil groups of different rank.

Now that we have a clear description of elliptic curves in terms of the Weierstrass form we can start to understand elliptic fibrations. As to each point of the base of the fibration B we want to associate an elliptic curve, we let the projective coordinates of the Weierstrass form and the coefficients f, g depend on the base. Since the base could be topologically non-trivial, rather than function, we should take f, g to be sections of line bundles over the base and we define an ambient five-fold which is a projective bundle over the base B

$$\mathbb{P}(\mathcal{O}, \mathcal{K}_B^{-2}, \mathcal{K}_B^{-3}). \tag{2.6.2}$$

The notation means that at each point of the base, the projective space associated to it has, as coordinates, sections of the bundle of constant functions over the base, $\mathcal{O}$, and sections of powers of the canonical bundle of the base, $\mathcal{K}_B$. Then if

$$x \in H^0(B, \mathcal{K}^2) \qquad y \in H^0(B, \mathcal{K}^3) \qquad z \in H^0, (B, \mathcal{O}), \tag{2.6.3}$$

the Weierstrass form

$$y^2 = x^3 + fxz^4 + gz^6 \tag{2.6.4}$$

describes an elliptic fibration over the base B. In order for the Weierstrass equation to have a homogeneous divisor class we require

$$f \in H^0(B, \mathcal{K}^4) \qquad g \in H^0(B, \mathcal{K}^6). \tag{2.6.5}$$

We can associate a divisor class to the coordinate hyperplanes

$$[x] = \alpha + 2c_1 \quad [y] = \alpha + 3c_1 \quad [z] = \alpha \tag{2.6.6}$$

where α is the hyperplane class of $\mathbb{P}^2$ and

$$\pi : Y \to B \quad c_1 = \pi^*(c_1(B)). \tag{2.6.7}$$

The assignments of the divisor classes follow from the the coordinates being sections of respective powers of the canonical bundle of the base (2.6.3). The Weierstrass equation is seen to be a section of $\mathcal{O}_{\mathbb{P}^2}(3)$ and its divisor class is

$$[Y] = 3\alpha + 6c_1. \tag{2.6.8}$$

In order to preserve $\mathcal{N} = 1$ supersymmetry in 4 dimensions, we require the elliptic fibration to be Calabi-Yau (this will become clear when discussing the F-Theory/M-Theory duality). A variety is Calabi-Yau if its canonical bundle is trivial, if it is Ricci-flat or, by the theorem proved by Yau, if its first Chern class vanishes. In order to determine the Chern class of our elliptically fibered variety, we are going to make use of the adjunction formula ([16]). Given an algebraic variety Y which is a subset of projective space $\mathbb{P}^n$ we can write down a short exact sequence of bundles, given by

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{\mathbb{P}^n}|_Y \longrightarrow \mathcal{N}_{\mathbb{P}^n/Y} \longrightarrow 0, \tag{2.6.9}$$

where $\mathcal{T}_{(\cdot)}$ is the tangent bundle of a variety and

$$\mathcal{N}_{\mathbb{P}^n/Y} \equiv \mathcal{T}_{\mathbb{P}^n}|_Y \,/\, \mathcal{T}_Y \tag{2.6.10}$$

is the normal bundle to Y in $\mathbb{P}^n$. Note that by definition of $\mathcal{N}_{\mathbb{P}^n/Y}$ the sequence (2.6.9) is exact. Given an exact sequence we can take the determinant line bundles of the bundles in the sequence to obtain another exact sequence

$$0 \longrightarrow \det \mathcal{T}_Y \longrightarrow \det \mathcal{T}_{\mathbb{P}^n}|_Y \longrightarrow \det \mathcal{N}_{\mathbb{P}^n/Y} \longrightarrow 0. \tag{2.6.11}$$

It follows from such an exact sequence that

$$\det \mathcal{T}_{\mathbb{P}^n}|_Y = \det \mathcal{T}_Y \otimes \det \mathcal{N}_{\mathbb{P}^n/Y}. \tag{2.6.12}$$

By definition the determinant line bundle of the tangent bundle to a variety is the canonical bundle to such variety $\mathcal{K}$, while since we are considering hypersurfaces in $\mathbb{P}^n$, the normal bundle $\mathcal{N}_{\mathbb{P}^n/Y}$ is a line bundle and

$$\det \mathcal{N}_{\mathbb{P}^n/Y} = \mathcal{N}_{\mathbb{P}^n/Y}. \tag{2.6.13}$$

Therefore we derive the adjunction formula

$$\mathcal{K}_Y = (\mathcal{K}_{\mathbb{P}^n} \otimes \mathcal{N}^*_{\mathbb{P}^n/Y})|_Y. \tag{2.6.14}$$

Equivalently in terms of divisor classes, this can be written as

$$[\mathcal{K}_Y] = ([\mathcal{K}_{\mathbb{P}^n}] + [Y])|_Y. \tag{2.6.15}$$

From the properties of Chern classes, it follows from the adjunction formula that the Chern class of Y can be expressed in terms of the ambient space X as

$$c(Y) = \frac{c(X)}{1+[Y]}\bigg|_Y. \tag{2.6.16}$$

It can be checked that in the case of the Weierstrass fibration $c(X) = 1 + 3\alpha + 6c_1 + \ldots$ and using the class of Y (2.6.8) we see that $c_1(Y)$ indeed vanishes

$$c(Y) = \frac{1 + 3\alpha + 6c_1 + \ldots}{1 + 3\alpha + 6c_1}\bigg|_Y = 1 + c_2(Y) + \ldots \tag{2.6.17}$$

The elliptic fibration becomes singular when the discriminant

$$\Delta = 4f^3 + 27g^2 \tag{2.6.18}$$

vanishes, which happens over a divisor in the base. Indeed we saw that an elliptic curve is defined only when the discriminant is non-vanishing. In the case of elliptic fibrations, the singularity can occur in the fiber (meaning only the tangent space to the fiber is degenerate) or in the whole variety. We will spend some time describing singularities of elliptic fibrations in the next section.

2.7 Singularities of Elliptic Fibrations

We saw that elliptic fibrations develop singularities whenever the discriminant Δ vanishes. From now on we will take the base of our fibration to be a Kahler three-fold by having in mind a reduction to 4 dimensions in the F-Theory set up. Therefore the fibration becomes singular over a codimension one locus in the base, that is over a complex surface. The correct physical interpretation is that a stack of 7-branes wrap such a divisor in the base: recall indeed that the complex structure-axiodilaton τ diverges at the location of the branes, and therefore the fibration degenerates.

Kodaira ([17]) classified all the possible singularities that an elliptic fibration over a complex 1-dimensional base can develop, and such a classification mostly holds for higher dimensional bases up to additional monodromies that we will discuss. In order to discuss the classification of singular elliptic fibrations, we will need to introduce some concepts in algebraic geometry. In particular, singular elliptic fibrations can be *resolved*, that is, a birational map can be found between them and a non-singular variety.

The main such procedure is called blowing up. Let us discuss the simple example of blowing up affine space $\mathbb{A}^n$ at a point to understand the main characteristics of this transform. Blowing up $\mathbb{A}^n$ at the origin means considering the variety given by

$$\{(x_1, \ldots, x_n), (y_1, \ldots, y_n) | x_i y_j = x_j y_i\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}. \tag{2.7.1}$$

Then we have a natural projection to the original variety given by

$$\pi : \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n, \tag{2.7.2}$$

which is birational. In particular we see that such a map is not well defined at the point where we blew up since

$$\pi^{-1}(0) \simeq \mathbb{P}^{n-1}, \tag{2.7.3}$$

that is we get the whole space of lines through the origin in the affine space $\mathbb{A}^n$. We call $\pi^{-1}(\mathbb{A}^n)$ the *total transform* of our affine space, while we call the closure of $\pi^{-1}(\mathbb{A}^n/\{0\})$ the *proper transform*. The exceptional locus $\pi^{-1}(0)$ is called the *exceptional divisor*. We can see why it can be a sensible thing to blow up a singular variety. Indeed, if a variety is singular at a point, the tangent space is degenerate at such point, but this does not have to be the case for the blown up variety, since the blown up point has been replaced by the exceptional divisor.

Let us look at the easiest hypersurface singularity, whose desingularization will be the template for more complex singularities. Let the A_1 singularity, the so-called simple double point, be described by the equation

$$P:\ x_1^2 + x_2^2 + x_3^2 = 0 \qquad \subset \qquad \mathbb{A}^3(x_1, x_2, x_3). \tag{2.7.4}$$

We immediately see that the hypersurface is singular at the origin since

$$P|_{(0,0,0)} = 0 \qquad dP|_{(0,0,0)} = 0, \tag{2.7.5}$$

where the first equation implies that the point $(x_1, x_2, x_3) = (0, 0, 0)$ does belong to the hypersurface, while the second equation implies that the tangent spce at that point is degenerate, and is equivalent to the condition that $\partial_i P|_{(0,0,0)} = 0$. In order to resolve such singularity we blow up the origin of the affine space as just explained, to find

$$\{x_1^2 + x_2^2 + x_3^2 = 0,\ x_i y_j = x_j y_i\} \subset \mathbb{A}^3 \times \mathbb{P}^2, \tag{2.7.6}$$

where $[y_0 : y_1 : y_2]$ are homogeneous coordinates on $\mathbb{P}^2$. We can then use the $\mathbb{C}^*$ action of $\mathbb{P}^2$ to fix $y_i \equiv 1$ on the patch U_i given by $y_i \neq 0$. Then we just substitute $x_j = y_j x_i$ in the equation of the singular hypersurface to find, on each patch

$$\begin{aligned} U_1 &:\ x_1^2(1 + y_2^2 + y_3^2) = 0 \\ U_2 &:\ x_2^2(y_1^2 + 1 + y_3^2) = 0 \\ U_3 &:\ x_3^2(y_1^2 + y_2^2 + 1) = 0 \end{aligned} \tag{2.7.7}$$

One can indeed check that the proper transform, given by the second branch in each patch (since setting $x_i = 0$ in U_i gives exactly the origin that we are blowing up), is not singular any more. One can also check through algebraic techniques that the Euler characteristic of the exceptional divisor is 2, that is we replaced the singular point on the hypersurface by a $\mathbb{P}^1 \simeq S^2$.

Resolving singularities in elliptic fibrations is for the most part similar to what we discussed so far. One looks at geometric loci which satisfy simultaneously the equations

$$Q|_{\mathbf{x}} = 0, \qquad dQ|_{\mathbf{x}} = 0, \tag{2.7.8}$$

where Q is the equation describing the fibration, and then repeatedly blows up the singular locus. An important concept is that since we start with a Calabi-Yau variety one might be worried that the resolved variety is not Calabi-Yau any more. This is a legitimate concern, which gives rise to the concept of crepant resolution, that is those resolutions which leave the canonical bundle of the variety unchanged (thus preserving the Calabi-Yau condition). It can be proved that the proper transform arising from the blowing up procedure preserves the Calabi-Yau condition (and so do *small resolutions* ([18])).

In order to classify the possible singularities of elliptic fibrations, Kodaira classified the exceptional divisors obtained by blowing up such singularities and studied how they intersect. It turns out that the singular locus is replaced by a chain of $\mathbb{P}^1$s which intersect according to an ADE classification. What that means is that the intersection matrices of the exceptional divisors are the Cartan matrices associated to Dynkin diagrams of type ADE. For example an A_1 singularity once resolved will have as exceptional divisor a single $\mathbb{P}^1$ which is represented in the affine Dynkin diagram by a single node. An A_2 singularity, once resolved, will give rise to two $\mathbb{P}_1$s that will intersect in two points - which translates into two nodes connected by a single line. And so on.

Kodaira then classified singularities in complex elliptic surfaces accordingly. Let us expand the coefficients of the Weierstrass form in a power series in a local coordinate z of the base, that is

$$f = \sum_i f_i z^i \qquad g = \sum_i g_i z^i. \tag{2.7.9}$$

Then the classification that Kodaira proposed associates ADE singularities to different vanishing orders of f, g and the discriminant Δ, as reported in Table 2.1.

In a similar spirit, Tate proposed an algorithm ([19]) which allows to determine the singularity of an elliptic curve/fibration starting from the vanishing orders of the coefficients $\{a_i\}$ of the so-called Tate form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \tag{2.7.10}$$

which is equivalent to the Weierstrass form. Such analysis, consisting of enhancing the vanishing order of the discriminant of the fibration by tuning the vanishing order of the coefficients $\{a_i\}$, was repeated for fibrations over higher dimensional bases ([20]) and it was found that there are some subtle differences compared to complex surfaces, such as singular fibers dual to F_4 and G_2 affine Dynkin diagrams, see Table 2.2. Moreover, in ([21]), a more thorough analysis was carried out so as to understand the possible ways to increase the vanishing order of the discriminant by using the fact that the coefficients of the Tate form belong to a unique factorization domain. Subtleties related to global behaviour of the sections might arise, and such an analysis was repeated in ([22]) for the elliptic fibration $\mathbb{P}^{112}[4]$. In Chapter 3 we will discuss instead Tate's algorithm applied to the elliptic fibration realized by a cubic equation embedded in $\mathbb{P}^2$.

$\mathcal{O}(f)$	$\mathcal{O}(g)$	$\mathcal{O}(\Delta)$	Fiber Type	Singularity Type
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 2.1: The classification of singular fibers depending on the vanishing orders of f, g and the discriminant Δ. Note both the Kodaira denomination of singular fibers and the corresponding ADE singularity type.

2.8 F-Theory and Dualities

Now that we discussed elliptic fibrations and their singularities, let us go back to F-Theory to discuss how the geometric properties of the fibration determine the physics of the compactifications. So far we discussed F-Theory as a technique to study type IIB compactificaitons in the presence of 7-branes. In order to understand the physics underlying such compactifications it will turn out to be useful to relate F-theory to M-Theory through a chain of dualities. Recall that M-Theory is the non-perturbative uplift of type IIA string theory where one dimension decompactifies to obtain an 11-dimensional theory whose low energy dynamics is 11-dimensional supergravity.

Let us consider M-Theory on $\mathbb{R}^{1,8} \times \mathbb{T}^2$, where we take the torus to be $\mathbb{T}^2 = S_A^1 \times S_B^1$ with complex structure τ. Then we follow the next steps:

- We let the radius R_A of S_A^1 go to zero, so to regain the perturbative regime of type IIA.
- We T-dualize using S_B^1 to obtain type IIB string theory on $\mathbb{R}^{(1,8)} \times \bar{S}_B^1$, where $\bar{S}_B^1$ has radius proportional to $1/R_B$.
- We let $R_B \to 0$ to decompactify to type IIB string theory. We obtain this way a

$\mathcal{O}(a_1)$	$\mathcal{O}(a_2)$	$\mathcal{O}(a_3)$	$\mathcal{O}(a_4)$	$\mathcal{O}(a_6)$	$\mathcal{O}(\Delta)$	Fiber Type	ADE Group
0	0	0	0	0	0	I_0	—
0	0	1	1	1	1	I_1	—
0	0	1	1	2	2	I_2	$SU(2)$
0	0	2	2	3	3	I_3^{ns}	unconven.
0	1	1	2	3	3	I_3^{s}	unconven.
0	0	n	n	$2n$	$2n$	I_{2k}^{ns}	$Sp(n)$
0	1	n	n	$2n$	$2n$	I_{2k}^{s}	$SU(2n)$
0	0	$n+1$	$n+1$	$2n+1$	$2n+1$	I_{2k+1}^{ns}	unconven.
0	1	n	$n+1$	$2n+1$	$2n+1$	I_{2k+1}^{s}	$SU(2n+1)$
1	1	1	1	1	2	II	—
1	1	1	1	2	3	III	$SU(2)$
1	1	1	2	2	4	IV^{ns}	unconven.
1	1	1	2	3	4	IV^{s}	$SU(3)$
1	1	2	2	3	6	I_0^{*ns}	G_2
1	1	2	2	4	6	I_0^{*ss}	$SO(7)$
1	1	2	2	4	6	I_0^{*s}	$SO(8)^*$
1	1	2	3	4	7	I_1^{*ns}	$SO(9)$
1	1	2	3	5	7	I_1^{*s}	$SO(10)$
1	1	3	3	5	8	I_2^{*ns}	$SO(11)$
1	1	3	3	5	8	I_2^{*s}	$SO(12)^*$
1	1	n	$n+1$	$2n$	$2n+3$	I_{2k-3}^{*ns}	$SO(4k+1)$
1	1	n	$n+1$	$2n+1$	$2n+3$	I_{2k-3}^{*s}	$SO(4k+2)$
1	1	$n+1$	$n+1$	$2n+1$	$2n+4$	I_{2k-2}^{*ns}	$SO(4k+3)$
1	1	$n+1$	$n+1$	$2n+1$	$2n+4$	I_{2k-2}^{*s}	$SO(4k+4)^*$
1	2	2	3	4	8	IV^{*ns}	F_4
1	2	2	3	5	8	IV^{*s}	E_6
1	2	3	3	5	9	III^*	E_7
1	2	3	4	5	10	II^*	E_8
1	2	3	4	6	12	non-min	—

Table 2.2: Tate's Algorithm with the vanishing orders of the coefficients $\{a_i\}$ of the Tate form. Note in particular the appearance of singular fiber absent from Kodaira's Classification arising from the higher dimension of the base.

duality between M-Theory on a torus of vanishing volume and type IIB which can be lifted to F-Theory.

Such duality can be extended fiberwise over the base of the fibration, therefore creating the set up of F-Theory. So we conclude that F-Theory on a Calabi-Yau fourfold is dual to M-Theory on the fourfold in the limit of vanishing fiber volume.

We can now use this duality to gain insight into the physics of F-Theory. We saw following Kodaira's classification that the singular elliptic fibration can be resolved to obtain a smooth variety, where the singular locus has been replaed by a tree Γ_i of $\mathbb{P}^1 s$ intersecting as in the dual affine Dynkin diagram. We can then reduce the C_3 form of M-Theory along these cycles to obtain the abelian gauge degrees of freedom A_i

$$A_i = \int_{\Gamma_i} C_3. \tag{2.8.1}$$

These degrees of freedom will form the Cartan subalgebra of our gauge symmetry. The non-abelian degrees of freedom are instead obtained by letting M2-branes wrap chains of $\mathbb{P}^1$s given by

$$S_{ij} = \Gamma_i \ \cup \Gamma_{i+1} \ \cup \cdots \cup \Gamma_j, \tag{2.8.2}$$

provided that Γ_k and Γ_{k+1} intersect. We obtain the degrees of freedom exactly to realize the gauge group indicated by the singularity type of the elliptic fibration. Therefore we see that F-Theory provides a completely geometric framework to study type IIB compactifications. In particular it lends itself easily to the realization of Grand Unified Theories since we can engineer 4-dimensional models by studying the singularities of elliptic fibrations. These are $\mathcal{N} = 1$ supersymmetric theories in 4 dimensions which have desirable phenomenological properties and that we will discuss in the next section.

We note here, without going into detail, that F-Theory also admits a duality to Heterotic string theory, although restricted to a smaller class of theories. In particular the duality states that F-Theory on an elliptically fibered K3 surface is dual to the Heterotic theory on T^2. We can extend this duality fiberwise to relate

$$\begin{gathered} \text{F-Theory on} \quad X : \, K3 \to B \\ \& \\ \text{Heterotic on} \quad Y : \, \mathbb{E} \to B, \end{gathered} \tag{2.8.3}$$

but we see that since the K3 surface is itself elliptically fibered

$$K3 : \; \mathbb{E} \to \mathbb{P}^1, \tag{2.8.4}$$

we must have on the F-Theory side a base B' which admits a $\mathbb{P}^1$ fibration, thus restricting the class of theories on which the duality can apply.

Let us now go back to the study of F-Theory compactifications and try to understand how the matter content and the couplings of the compactifications are encoded in the geometry of the fibration. So far we saw that the codimension one singularity structure of the fibration determines the gauge group of the 4-dimensional theory. Nevertheless, since the base of our fibration is not just one complex dimensional, we see that different phenomena might happen in higher codimension. In particular for the case of compactifications to 4 dimensions, the base is a complex threefold and we have codimension 2 and codimension 3 singularity at our disposal (that is singularities specified locally by respectively 2 or 3 equations).

In brane set ups, matter is found at the intersection of (stacks of) branes as open string excitations stretching between the two branes. In F-Theory context we might have m 7-branes wrapping a four cycle M_4 in the base and n 7-branes wrapping a different four cycle M_4'. Then since codimensions add, we see that generically the two stacks of branes intersect over a complex curve Σ in the base, that is

$$\begin{cases} m \text{ 7-branes on } \mathbb{R}^{1,3} \times M_4 \\ n \text{ 7-branes on } \mathbb{R}^{1,3} \times M_4' \end{cases} \xrightarrow{\cap} \mathbb{R}^{1,3} \times \Sigma \subset \mathbb{R}^{1,3} \times M_6. \tag{2.8.5}$$

If gauge symmetrise G_a and G_b are associated to the two stacks of 7-branes, and at the intersection we have an enhanced G_{ab} symmetry we find matter in the representation (R_x, U_x) under the breaking of the adjoint of $G_{ab} \to G_a \times G_b$ ([14])

$$\begin{aligned} G_{ab} &\to G_a \times G_b \\ ad_{G_{ab}} &\to (ad_{G_a}, \mathbf{1}) \oplus (\mathbf{1}, ad_{G_b}) \oplus \sum (R_x, U_x). \end{aligned} \tag{2.8.6}$$

For example enhancements from $SU(5)$ to $SO(10)$ allow matter in the $\mathbf{10}$ ($\bar{\mathbf{10}}$) following the decomposition of the adjoint of $SO(10)$

$$\mathbf{45} \to \mathbf{24}_0 + \mathbf{10}_2 + \bar{\mathbf{10}}_{-2} + \mathbf{1}_0. \tag{2.8.7}$$

2.9 Abelian Symmetries in F-Theory

It should now be clear that F-Theory is an excellent framework for studying string compactifications and geometrically engineer supersymmetric 4-dimensional theories. This turned out to be useful in order to realise Grand Unified Theories in which the fundamental gauge interactions of the Standard Model are united in a single gauge group, typically $SU(5)$ or $SO(10)$, which then at some energy scale (around 10^{16} GeV) breaks to the Standard Model gauge group. Such theories captured a lot of attention in the past since it was possible to embed the Standard Model gauge group as a maximal subgroup of a unified group, thus providing the unification of the known interactions which was confirmed by the running of the couplings under the RG flow. Even though supersymmetry potentially allows to solve problems afflicting the Standard Model (Dark Matter, Hierarchy Problem, etc), Grand Unified Theories suffer themselves from phenomenological difficulties.

In particular the adjoint of $SU(5)$ under the breaking to the Standard Model gauge group

$$SU(5) \to SU(3) \times SU(2) \times U(1) \tag{2.9.1}$$

decomposes as follows

$$\mathbf{24} \to (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{3},\bar{\mathbf{2}})_{-5} \oplus (\bar{\mathbf{3}},\mathbf{2})_5, \tag{2.9.2}$$

where the first three terms reproduces correctly the content of the Standard Model, but we see the appearance of additional gauge bosons. The main threat comes from the fact that such bosons are able to induce proton decays operators. Protons have experimentally been shown to possess a half-life which is greater than the life of the universe, so we see that the presence of proton decay operators in Grand Unified Theories presents a considerable problem to the unification program.

One way out of this impasse was found to rely on the existence of additional abelian factors in the unified gauge group. Ideally one would like to find an additional factor $U(1)^n$ such that the $U(1)$ charges would allow operators such as the Top Yukawa coupling $\mathbf{10}_M\mathbf{10}_M\mathbf{5}_H$, but prevent the proton decay operator $\mathbf{10}_M\mathbf{10}_M\mathbf{5}_M$ by deeming it not gauge invariant.

We saw that in F-Theory we can only engineer theories with a gauge group of ADE type and there does not seem to be room for additional abelian factors. It turns out that the correct way to solve this problem is to look at elliptic fibrations which admit extra rational sections. Recall that the Weierstrass model embedded in $\mathbb{P}^{123} = [z : x : y]$ admitted one rational point given by

$$[1 : 1 : 0] \in \ \{y^2 = x^3 + fxz^4 + gz^6\}, \tag{2.9.3}$$

and by fibering over the base this implied the existence of a global section of the elliptic fibration. Consider now the existence of additional rational points on an elliptic curve and therefore the existence of additional sections of the elliptic fibration. Then such sections are divisors in the Calabi-Yau fourfold and we can consider their Poincare Dual (1,1)-forms ω^i. It is clear that we can use these to reduce the C_3 form in the M-Theory framework to obtain abelian degrees of freedom A_i

$$C_3 = A_i \wedge \omega^i + \ldots . \tag{2.9.4}$$

We therefore find a gauge field for an abelian $U(1)$ factor for each extra section of the elliptic fibrations. Now that we know how to engineer additional abelian gauge degrees of freedom in F-Theory, let us see how we can calculate the $U(1)$ charges of matter arising from codimension two singularities. We define the Shioda map ([23])

$$\mathcal{S} : \text{Mordell-Weil Group} \longrightarrow H^{1,1}(Y_4), \tag{2.9.5}$$

which associates to each section generating the Mordell-Weill group a divisor of the elliptic Calabi-Yau Y_4. The $U(1)$ charge of a matter curve Σ associated to a section σ_i is found to be the intersection $\Sigma \cdot_{Y_4} \mathcal{S}(\sigma_i)$.

In order to have an elliptic curve with an extra rational point (that will become an extra section upon fibering over the base) we need to embed it in the weighted projective space $\mathbb{P}^{112}[w : x : y]$ by the equation ([23])

$$c_0 w^4 + c_1 w^3 x + c_2 w^2 x^2 + c_3 w x^3 = y^2 + b_0 x^2 y, \tag{2.9.6}$$

and the two rational points are seen to be

$$\sigma_1 = [0 : 1 : 0] \qquad \sigma_2 = [0 : 1 : -b_0]. \tag{2.9.7}$$

The study of the possible singularities occurring in an elliptic fibration with an extra rational section was carried out in ([22]). Similarly an elliptic curve with two extra rational sections can be realized as a cubic equation in projective space $\mathbb{P}^2$ ([1, 24–26])

$$s_1 w^3 + s_2 w^2 x + s_3 w x^2 + s_5 w^2 y + s_6 w x y + s_7 x^2 y + s_8 w y^2 + s_9 x y^2 = 0, \tag{2.9.8}$$

with the three rational points

$$\sigma_1 = [0 : 0 : 1] \qquad \sigma_2 = [0 : 1 : 0] \qquad \sigma_3 = [0 : s_9 : -s_7]. \tag{2.9.9}$$

In Chapter 3 we will study the possible singularities of such an elliptic fibration through the application of Tate's algorithm.

Chapter 3

Tate's Algorithm for F-theory GUTs with two U(1)s

As outlined in Chapter 2, the compactification of F-Theory on elliptically fibered Calabi-Yau manifolds has proven to be a successful framework to realize supersymmetric non-abelian gauge theories, in particular Grand Unified Theories (GUTs) ([27–29]). Although GUTs are an appealing framework for supersymmetric model building[1], it is well known that they can suffer from fast proton decay, which, however, can be obviated by having additional discrete or continuous symmetries. In this chapter we consider F-theory compactifications that give rise to GUTs with two additional $U(1)$s, which can potentially be used to suppress certain proton decay operators[2]. In F-theory abelian gauge factors have their genesis in geometric properties of the compactification manifold, namely in the existence of additional rational sections of the elliptic fibration. We carry out a systematic procedure to constrain which such fibrations can give rise to gauge groups $G \times U(1)^2$.

It has been known for many years that abelian gauge symmetries in F-theory are characterized by the Mordell-Weil group of the elliptically fibered Calabi-Yau compactification space ([8,9]), which is the group formed by the rational sections of the fibration. In recent years abelian gauge factors have been much studied in the context of 4-dimensional GUTs arising from F-theory compactifications. In local F-theory models $U(1)$s have a realization in terms of factored spectral covers as shown in ([37–44]). Global models with one $U(1)$ were studied in ([22,23,45–52]), however phenomenologically one $U(1)$ factor is not sufficient to forbid all dangerous couplings ([53]). It is then well moti-

[1] See ([14,30,31]) for some nice reviews of GUT model building in F-theory.

[2] Discrete symmetries have been studied in local and global F-theory model building in, e.g. ([32–36]).

vated to consider elliptic fibrations with multiple $U(1)$ factors, the constuction of which was initiated in ([1, 24–26, 54–58]), with the realization of the $SU(5) \times U(1)^2$ models in these papers primarily based on constructions from toric tops ([59]).

It is natural to ask whether there is a systematic way to explore the full range of possible low-energy theories with two additional abelian gauge factors which have an F-theory realization. One approach to address this question is to apply Tate's algorithm ([19–21]) to elliptic fibrations with two additional rational sections. This is the approach that we take in this chapter and indeed we show that there is a large class of new elliptic fibrations with phenomenologically interesting properties not seen from the top constructions. While Tate's algorithm is a comprehensive method to obtain the form of any elliptic fibration with two rational sections there is a caveat that it is sometimes difficult in practice to proceed with the algorithm without making simplifications at the cost of generality.

The starting point for the application of Tate's algorithm in this context is the realization of the elliptic fiber as a cubic in $\mathbb{P}^2$ ([1, 24–26]). Tate's algorithm involves the study of the discriminant of this cubic equation, which captures the information about the singularities of the fiber. The singular fibers of an elliptic surface were classified by Kodaira ([60, 61]) and Néron ([62]), and they belong to an ADE-type classification; Tate's algorithm is a systematic procedure to determine the type of singular fiber. The ADE type of the singular fiber determines the non-abelian part of the gauge symmetry.

Tate's algorithm was applied to the Weierstrass form for an elliptic fibration where there are generically no $U(1)$s in ([20,21]), and in ([22]) to the quartic equation in $\mathbb{P}^{(1,1,2)}$ which realizes a single $U(1)$ ([23]). The application of the algorithm to the cubic in $\mathbb{P}^2$ will constrain the form of the fibrations which realize a $G \times U(1)^2$ symmetry, for some non-abelian gauge group G, which are phenomenologically interesting for model building.

As a result of Tate's algorithm we find a collection of elliptic fibrations which realize the gauge symmetry $SU(5) \times U(1)^2$ where the non-abelian symmetry is the minimal simple Lie group containing the Standard Model gauge group. The fibrations found encompass all of the $SU(5)$ models with two $U(1)$s in the literature which we are aware of, and includes previously unknown models which, in many cases, have exciting phenomenological features, such as having multiple, differently charged, **10** matter curves. We also determine fibrations that lead to E_6 and $SO(10)$ gauge groups with two $U(1)$s.

Our results are not restricted to F-theoretic GUT model building, and we hope that they are also useful in other areas of F-theory, for example in direct constructions of the Standard Model ([63, 64]), in the determination of the network of resolutions of elliptic fibrations ([65–69]), or in the recent relationship drawn between elliptic fibrations with

$U(1)$s and genus one fibrations with multisections ([70–72]).

In section 3.1 we present a summary where we highlight the fibrations found in the application of Tate's algorithm to the cubic equation, up to fibers realizing $SU(5)$. We also present a table of a particularly nice kind of realizations for Kodaira fibers I_n and I_n^*. In section 3.2 we recap the embedding of the elliptic fibration as a cubic hypersurface in a $\mathbb{P}^2$ fibration and give details of the resolution and intersection procedures. Section 3.3 contains Tate's algorithm proper, up to the I_5, or $SU(5)$, singular fibers. In section 3.4 the $U(1)$ charges of the various **10** and **5** matter curves that appear in the models from the $SU(5)$ singular fibers are determined. In section 3.5 Tate's algorithm is continued from where it was left off in section 3.3 and we obtain fibrations that have a non-abelian component corresponding to an exceptional Lie algebra.

3.1 Overview and Summary

For the reader's convenience, the key results are summarized in this section. For those interested simply in the new $SU(5)$ models we refer to section 3.4.

An elliptic fibration with two additional rational sections, which gives rise to a gauge theory with two additional $U(1)$s, can be realized as a hypersurface in a $\mathbb{P}^2$ fibration, as in ([1, 24–26]), given by the equation

$$\mathfrak{s}_1 w^3 + \mathfrak{s}_2 w^2 x + \mathfrak{s}_3 w x^2 + \mathfrak{s}_5 w^2 y + \mathfrak{s}_6 w x y + \mathfrak{s}_7 x^2 y + \mathfrak{s}_8 w y^2 + \mathfrak{s}_9 x y^2 = 0 \,, \tag{3.1.1}$$

where $[w : x : y]$ are projective coordinates on the $\mathbb{P}^2$. This fibration has three sections which have projective coordinates

$$\Sigma_0 : [0 : 0 : 1] \,, \quad \Sigma_2 : [0 : 1 : 0] \,, \quad \Sigma_1 : [0 : \mathfrak{s}_9 : -\mathfrak{s}_7] \,. \tag{3.1.2}$$

The application of Tate's algorithm involves enhancing the singularity of this elliptic fibration, where the particular enhancements are determined by the discriminant. As the coefficients of the fibration are sections of holomorphic line bundles over the base, one can look at an open neighbourhood around the singular locus in the base with coordinate z such that the singular locus is above $z = 0$, and consider the expansion in the coordinate z of the $\mathfrak{s}_i$

$$\mathfrak{s}_i = \sum_{j=0}^{\infty} s_{i,j} z^j \,. \tag{3.1.3}$$

Often the pertinent information from the equation (3.1.1) is just the vanishing orders of the $\mathfrak{s}_i$ in z, which we will refer to through

$$n_i = \mathrm{ord}_z(\mathfrak{s}_i) \,. \tag{3.1.4}$$

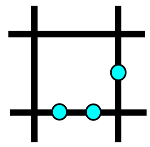
A shorthand for the equation will be the tuple of positive integers $(n_1, n_2, n_3, n_5, n_6, n_7, n_8, n_9)$ representing the vanishing orders. It will not always be possible to express a fibration just through a set of vanishing orders, but there will also be non-trivial relations among the coefficients of the equation. We will refer to fibrations of this form as *non-canonical models*. This will be the result of solving in full generality the polynomials which appear in the discriminant as a necessary condition for enhancing the singular fiber. In particular the fact that the coefficients of our fibration belong to a unique factorization domain ([21, 73]) will be used. Schematically we will refer to these fibrations via the shorthand notation

$$I_{nc^i} : \quad \left\{ \begin{array}{c} (n_1, n_2, n_3, n_5, n_6, n_7, n_8, n_9) \\ [s_{1,n_1}, s_{2,n_2}, s_{3,n_3}, s_{5,n_5}, s_{6,n_6}, s_{7,n_7}, s_{8,n_8}, s_{9,n_9}] \end{array} \right\}, \tag{3.1.5}$$

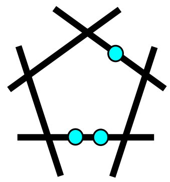
where the term in square brackets denotes any specialization of the leading non-vanishing coefficients in the expansion of the $\mathfrak{s}_i$, and the I represents the Kodaira fiber type. Often, for ease of reading, a dash will be inserted to indicate that a particular coefficient is unspecialized. The exponent of the index nc will signal how many non-canonical enhancements of the discriminant were used in order to obtain the singular fiber, that is, how many times solving a polynomial in the discriminant did not require just setting some of the expansion coefficients to zero, but also some additional cancellation.

There is a last piece of notation that needs to be explained before the results can be presented. Since the elliptic fibration has three sections, it will be seen in section 3.3, where the algorithm is studied in detail, that the discriminant will reflect the fact that the sections can intersect the components of the resolved fiber in multiple different ways. Thus, a number of (non-)canonical forms for each Kodaira singular fiber will be obtained depending on which fiber component each of the sections intersects. To represent this, denote by $I_n^{(012)}$ the case where all the three sections intersect the same fiber component, and then introduce separation of the sections by means of the notation $I_n^{(0|^n1|^m2)}$, where the number of slashes will signal the distance between the fiber components that the corresponding sections intersect. Consider the two examples:

- $I_{4,nc^2}^{s(01|2)}$ will represent a Kodaira singular fiber I_4, obtained through two non-canonical enhancements of the discriminant. The sections Σ_0, Σ_2 will intersect one of the fiber components, while Σ_1 will sit on an adjacent fiber component (i.e. one which intersects the previous component). Depicting the $\mathbb{P}^1$ components of the singular fiber as lines, and the sections as nodes, the fiber $I_4^{s(01|2)}$ can be represented by the diagram



- $I_{5,nc^3}^{s(01||2)}$ will represent an I_5 found upon imposing non-canonical conditions on the coefficients of our equation three times, such that the fiber component intersected by Σ_0 and Σ_1 does not intersect the component that Σ_2 intersects. This $I_5^{s(01||2)}$ is represented pictorially as



We refer to section 3.5 for more details about the notation for representing singular fibers corresponding to other types of Kodaira singular fibers.

All of the fibers found and determined are presented in the following summary tables, where the fibers are grouped first by the Kodaira type and then by the degree of canonicality:

- In table 3.1 we list the singular fibers up to vanishing order $\mathrm{ord}_z(\Delta) = 3$. These include fibers of type I_1,I_2,I_3, II, and III.

- In table 3.2 we list the singular fibers at vanishing order $\mathrm{ord}_z(\Delta) = 4$. These include both type I_4 and type IV Kodaira fibers.

- In table 3.3 we list the I_5 singular fibers.

For each of the I_5 singular fibers obtained through the algorithm the $U(1)$ charges are calculated and the results are presented in section 3.4, along with the comparison with the $U(1)$ charges of the known $SU(5)$ toric tops ([1, 24, 51, 54, 59]).

Tate-like (that is, canonical) forms for generic Kodaira singular fibers were also determined and they are presented in table 3.4. Appendix A.3 includes explicit details of the resolutions of these forms.

Singular Fiber	Vanishing Orders and Non-canonical Data
$I_1^{(012)}$	$(1,1,0,1,0,0,0,0)$
$I_2^{(012)}$	$(2,1,0,1,0,0,0,0)$
$I_2^{(01\|2)}$	$(1,1,1,0,0,0,0,0)$
$I_{2,nc}^{(1\|02)}$	$(1,1,0,1,0,0,0,0)$ $[-,-,\sigma_2\sigma_5,-,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_1\sigma_2,\sigma_3\sigma_4,\sigma_1\sigma_3]$
$II_{nc}^{(012)}$	$(1,1,0,1,0,0,0,0)$ $[-,-,\mu\sigma_1^2,-,2\mu\sigma_3\sigma_8,-,\mu\sigma_8^2,-]$
$I_3^{ns(012)}$	$(3,2,0,2,0,0,0,0)$
$I_3^{s(01\|2)}$	$(2,1,1,1,0,0,0,0)$
$I_3^{s(0\|1\|2)}$	$(1,1,1,1,0,0,1,0)$
$I_{3,nc}^{s(012)}$	$(3,1,0,1,0,0,0,0)$ $[-,\sigma_1\sigma_2,\sigma_2\sigma_5,\sigma_1\sigma_3,\sigma_2\sigma_4+\sigma_3\sigma_5,-,\sigma_3\sigma_4,-]$
$I_{3,nc}^{s(01\|2)}$	$(2,1,1,0,0,0,0,0)$ $[-,\sigma_1\sigma_2,\sigma_1\sigma_3,\sigma_2\sigma_5,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_3\sigma_4,-,-]$
$I_{3,nc}^{s(02\|1)}$	$(2,1,0,1,0,0,0,0)$ $[-,-,\sigma_2\sigma_5,-,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_1\sigma_2,\sigma_3\sigma_4,\sigma_1\sigma_3]$
$I_{3,nc}^{s(0\|1\|2)}$	$(1,1,1,0,0,0,0,0)$ $[-,-,-,\sigma_2\sigma_5,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_3\sigma_4,\sigma_1\sigma_2,\sigma_1\sigma_3]$
$III_{nc}^{(012)}$	$(2,1,0,1,0,0,0,0)$ $[-,-,\mu\sigma_3^2,-,2\mu\sigma_3\sigma_8,-,\mu\sigma_8^2,-]$
$III_{nc}^{(01\|2)}$	$(1,1,1,0,0,0,0,0)$ $[-,-,-,\mu\sigma_5^2,2\mu\sigma_5\sigma_7,\mu\sigma_7^2,-,-]$
$III_{nc^2}^{(02\|1)}$	$(1,1,0,1,0,0,0,0)$ $[-,-,\xi_2^2\xi_4,-,2\xi_2\xi_3\xi_4,\sigma_1\xi_2,\xi_3^2\xi_4,\sigma_1\xi_3]$

Table 3.1: Singular fibers where $\mathrm{ord}_z(\Delta) \leq 3$.

Singular Fiber	Vanishing Orders and Non-canonical Data
$I_4^{ns(012)}$	$(4,2,0,2,0,0,0,0)$
$I_4^{s(01\|2)}$	$(3,2,1,1,0,0,0,0)$
$I_4^{ns(01\|\|2)}$	$(2,2,2,0,0,0,0,0)$
$I_4^{s(0\|1\|2)}$	$(2,1,1,1,0,0,1,0)$
$I_{4,nc}^{s(012)}$	$(4,2,0,2,0,0,0,0)$ $[-,-,\sigma_1\sigma_3,-,\sigma_1\sigma_2+\sigma_3\sigma_4,-,\sigma_2\sigma_4,-]$
$I_{4,nc}^{s(01\|2)}$	$(2,1,1,1,0,0,0,0)$ $[\sigma_3\sigma_4,\sigma_1\sigma_3,-,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_1\sigma_2,-,\sigma_2\sigma_5,-]$
$I_{4,nc}^{s(02\|1)}$	$(3,2,0,2,0,0,0,0)$ $[-,-,\sigma_2\sigma_5,-,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_1\sigma_2,\sigma_3\sigma_4,\sigma_1\sigma_3]$
$I_{4,nc}^{s(01\|\|2)}$	$(2,2,2,0,0,0,0,0)$ $[-,-,-,\sigma_1\sigma_3,\sigma_1\sigma_2+\sigma_3\sigma_4,\sigma_2\sigma_4,-,-]$
$I_{4,nc}^{s(01\|\|2)}$	$(2,1,1,1,0,0,0,0)$ $[-,\sigma_1\sigma_2,\sigma_1\sigma_3,-,\sigma_2\sigma_4,\sigma_3\sigma_4,-,-]$
$I_{4,nc}^{s(1\|0\|2)}$	$(1,1,1,1,0,0,1,0)$ $[\sigma_2\sigma_5,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_3\sigma_4,-,\sigma_1\sigma_2,\sigma_1\sigma_3,-,-]$
$I_{4,nc^2}^{s(02\|1)}$	$(3,2,0,2,0,0,0,0)$ $[-,-,\sigma_3\xi_1\xi_2,-,\sigma_2\xi_1\xi_2+\sigma_3\xi_1\xi_3,\xi_2\xi_4,\xi_1\xi_3\sigma_2,\xi_3\xi_4]$
$I_{4,nc^2}^{s(0\|1\|2)}$	$(1,1,1,0,0,0,0,0)$ $[\xi_3\xi_4,\xi_2\xi_4+\xi_3\xi_5,\xi_2\xi_5,\xi_3\sigma_5,\xi_3\sigma_4+\xi_2\sigma_5,\xi_3\sigma_4,\sigma_1\xi_3,\sigma_1\xi_2]$
$I_{4,nc^2}^{s(1\|0\|2)}$	$(1,1,1,0,0,0,0,0)$ $[\xi_3\xi_4,\xi_2\xi_4+\xi_3\xi_5,\xi_2\xi_5,\sigma_2\xi_1\xi_3,\sigma_2\xi_1\xi_3,\sigma_2\xi_1\xi_2+\sigma_3\xi_1\xi_3,\sigma_3\xi_1\xi_2,\sigma_1\sigma_2,\sigma_1\sigma_3]$
$IV^{s(01\|2)}$	$(2,1,1,1,1,0,0,0)$
$IV^{s(0\|1\|2)}$	$(1,1,1,1,1,0,1,0)$
$IV_{nc^2}^{ns(012)}$	$(2,1,0,1,0,0,0,0)$ $[-,\xi_1\xi_3,\mu\xi_3^2,\xi_1\xi_2,2\mu\xi_2\xi_3,-,\mu\xi_2^2,-]$
$IV_{nc^2}^{s(01\|2)}$	$(1,1,1,0,0,0,0,0)$ $[\xi_2\xi_5,\xi_2\xi_4+\xi_3\xi_5,\xi_3\xi_4,\mu\xi_2^2,2\mu\xi_2\xi_3,\mu\xi_3^2,-,-]$
$IV_{nc^2}^{s(02\|1)}$	$(2,1,0,1,0,0,0,0)$ $[-,-,\xi_4\xi_2^2,-,2\xi_2\xi_3\xi_4,\sigma_1\xi_2,\xi_4\xi_3^2,\sigma_1\xi_3]$
$IV_{nc^2}^{s(0\|1\|2)}$	$(1,1,1,0,0,0,0,0)$ $[-,-,-,\mu\xi_2^2,2\mu\xi_2\xi_3,\mu\xi_3^2,\xi_2\xi_4,\xi_3\xi_4]$
$IV_{nc^3}^{s(012)}$	$(2,1,0,1,0,0,0,0)$ $[\delta_2\delta_4,\xi_3(\delta_1\delta_2+\delta_3\delta_4),\delta_1\delta_3\xi_3^2,\xi_2(\delta_1\delta_2+\delta_3\delta_4),2\delta_1\delta_3\xi_2\xi_3,-,\delta_1\delta_3\xi_2^2,-]$

Table 3.2: Singular fibers where $\mathrm{ord}_z(\Delta)=4$.

Singular Fiber	Vanishing Orders and Non-canonical Data
$I_5^{ns(012)}$	$(5,3,0,3,0,0,0,0)$
$I_5^{s(01\|2)}$	$(4,2,1,2,0,0,0,0)$
$I_5^{s(01\|\|2)}$	$(3,2,2,1,0,0,0,0)$
$I_5^{s(0\|1\|2)}$	$(3,2,1,1,0,0,1,0)$
$I_5^{s(0\|1\|\|2)}$	$(2,2,2,1,0,0,1,0)$
$I_{5,nc}^{s(012)}$	$(5,2,0,2,0,0,0,0)$ $[-,\sigma_1\sigma_2,\sigma_2\sigma_5,\sigma_1\sigma_3,\sigma_2\sigma_4+\sigma_3\sigma_5,-,\sigma_3\sigma_4,-]$
$I_{5,nc}^{s(01\|2)}$	$(3,2,1,1,0,0,0,0)$ $[\sigma_2\sigma_5,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_3\sigma_4,\sigma_1\sigma_2,\sigma_1\sigma_3,-,-,-]$
$I_{5,nc}^{s(02\|1)}$	$(4,2,0,2,0,0,0,0)$ $[-,-,\sigma_3\sigma_4,-,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_1\sigma_3,\sigma_2\sigma_5,\sigma_1\sigma_2]$
$I_{5,nc}^{s(0\|1\|2)}$	$(2,1,1,1,0,0,1,0)$ $[\sigma_1\sigma_3,\sigma_1\sigma_2,-,\sigma_3\sigma_4,\sigma_2\sigma_4,-,-,-]$
$I_{5,nc}^{s(1\|0\|2)}$	$(3,2,1,1,0,0,0,0)$ $[-,-,-,-,\sigma_1\sigma_2,\sigma_1\sigma_3,\sigma_2\sigma_4,\sigma_3\sigma_4]$
$I_{5,nc}^{s(0\|1\|\|2)}$	$(2,2,2,0,0,0,0,0)$ $[-,-,-,\sigma_2\sigma_5,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_3\sigma_4,\sigma_1\sigma_2,\sigma_1\sigma_3]$
$I_{5,nc}^{s(0\|1\|\|2)}$	$(2,1,1,1,0,0,1,0)$ $[-,\sigma_1\sigma_2,\sigma_1\sigma_3,-,\sigma_2\sigma_4,\sigma_3\sigma_4,-,-]$
$I_{5,nc^2}^{s(02\|1)}$	$(4,2,0,2,0,0,0,0)$ $[-,-,\sigma_3\xi_2,-,\sigma_2\xi_2+\sigma_3\xi_3,\xi_2\xi_4,\sigma_2\xi_3,\xi_3\xi_4]$
$I_{5,nc^2}^{s(01\|\|2)}$	$(2,1,1,1,0,0,0,0)$ $[\xi_3\xi_4,\sigma_2\xi_3,\sigma_3\xi_3,\xi_2\xi_4+\xi_3\xi_5,\sigma_2\xi_2,\sigma_3\xi_2,\xi_2\xi_5,-]$
$I_{5,nc^2}^{s(01\|\|2)}$	$(2,2,2,0,0,0,0,0)$ $[\xi_3\xi_4,\xi_2\xi_4+\xi_3\xi_5,\xi_2\xi_5,\sigma_3\xi_3,\sigma_2\xi_3+\sigma_3\xi_2,\sigma_2\xi_2,-,-]$
$I_{5,nc^2}^{s(1\|0\|2)}$	$(2,1,1,1,0,0,0,0)$ $[\sigma_3\sigma_4,\sigma_3\xi_1\xi_3,-,\sigma_2\sigma_4+\sigma_3\xi_1\xi_2,\sigma_2\xi_1\xi_3,\xi_3\xi_4,\sigma_2\xi_1\xi_2,\xi_2\xi_4]$
$I_{5,nc^2}^{s(0\|1\|\|2)}$	$(2,1,1,1,0,0,0,0)$ $[-,\sigma_1\xi_3,\sigma_1\xi_2,-,\sigma_4\xi_3,\sigma_4\xi_2,\xi_3\xi_4,\xi_2\xi_4]$
$I_{5,nc^2}^{s(0\|2\|\|1)}$	$(1,1,1,1,0,0,1,0)$ $[\xi_3\xi_4\xi_5\xi_6,\sigma_4\xi_5\xi_6+\sigma_3\xi_3\xi_4,\sigma_3\sigma_4,\xi_3\xi_5\xi_7+\xi_4\xi_6\xi_8,\xi_1\xi_3\xi_5\xi_6,\sigma_3\xi_1\xi_3,\xi_7\xi_8,\xi_1\xi_6\xi_8]$
$I_{5,nc^3}^{s(0\|1\|\|2)}$	$(1,1,1,0,0,0,0,0)$ $[\xi_3\delta_3\delta_4,\delta_4(\delta_3\xi_2+\delta_2\xi_3),\xi_2\delta_2\delta_4,\xi_3\delta_1\delta_3,\delta_1(\delta_2\xi_3+\delta_3\xi_2),\delta_1\delta_2\xi_2,\sigma_1\xi_3,\sigma_1\xi_2]$

Table 3.3: Singular fibers where $\mathrm{ord}_z(\Delta)=5$.

3.2 Setup

In this section the general setup for the discussion of singular elliptic fibrations with a rank two Mordell-Weil group is provided. First it is explained in more detail that such a fibration can be embedded into a $\mathbb{P}^2$ fibration via a cubic hypersurface equation. This is done in section 3.2.1. In section 3.2.2 the symmetries of this cubic equation are detailed and it is demonstrated how they lead to a redundancy of singular fiber types. Some constraints are chosen, listed at the head of section 3.2.2, to eliminate this redundancy. All the properties of the construction used in the resolution and study of the singular fibers found are documented in section 3.2.3.

3.2.1 Embedding

By the algebro-geometric construction in ([1, 23–26]), an elliptic fibration with rank two Mordell-Weil group can be embedded into a $\mathbb{P}^2$ fibration by the hypersurface equation

$$\mathfrak{s}_1 w^3 + \mathfrak{s}_2 w^2 x + \mathfrak{s}_3 w x^2 + \mathfrak{s}_5 w^2 y + \mathfrak{s}_6 wxy + \mathfrak{s}_7 x^2 y + \mathfrak{s}_8 w y^2 + \mathfrak{s}_9 x y^2 = 0\,, \tag{3.2.1}$$

as seen in the previous section. Some explanation of this construction is given in appendix A.4. Here $[w : x : y]$ are the projective coordinates of the fibration and the $\mathfrak{s}_i$ are elements of the base coordinate ring, R. It can be seen that this has three marked points, where w, x, and y take values in the fraction field, K, associated to R. Specifically the three marked points are

$$[0:0:1]\,, \quad [0:1:0]\,, \quad [0:\mathfrak{s}_9:-\mathfrak{s}_7]\,, \tag{3.2.2}$$

which we label as Σ_0, Σ_2, and Σ_1 respectively.

We will work in an open neighbourhood in the base, around the singular locus, which has coordinate z such that the singular locus will occur at the origin of this open neighbourhood. In such a local patch we can specify the $\mathfrak{s}_i$ as expansions in z,

$$\mathfrak{s}_i = \sum_{j=0}^{\infty} s_{i,j} z^j\,. \tag{3.2.3}$$

We also introduce the simplifying notation

$$\mathfrak{s}_{i,k} = \sum_{j=k}^{\infty} s_{i,j} z^{j-k}\,. \tag{3.2.4}$$

3.2.2 Symmetries and Lops

In this section note is made of the symmetries inherent in the cubic equation (3.1.1), and a strategy is devised to remove the redundant multiplicity of fiber types that occurs due

Fiber Type	Gauge Group	Vanishing Orders of the Coefficients								$\mathcal{O}(\Delta)$	Restrictions
		$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_5$	$\mathfrak{s}_6$	$\mathfrak{s}_7$	$\mathfrak{s}_8$	$\mathfrak{s}_9$		
$I_{2k+1}^{s(0\|^n1\|^m2)}$	$SU(2k+1)$	$2k+1-(n+m)$	$k-n$	m	$k+1-m$	0	0	n	0	$2k+1$	$m+n \leq k$
$I_{2k+1}^{s(0\|^n1\|^m2)}$	$SU(2k+1)$	$2k+1-(n+m)$	m	m	n	0	0	n	0	$2k+1$	$m+n \leq \lfloor \frac{2}{3}(2k+1) \rfloor$
$I_{2k}^{s(0\|^n1\|^m2)}$	$SU(2k)$	$2k-(n+m)$	$k-n$	m	$k-m$	0	0	n	0	$2k$	$m+n \leq k, \quad m < k$
$I_{2k}^{s(0\|^n1\|^m2)}$	$SU(2k)$	$2k-(n+m)$	m	m	n	0	0	n	0	$2k$	$m+n \leq \lfloor \frac{4}{3}k \rfloor$
$I_{2k}^{ns(01\|^n2)}$	k	k	k	0	0	0	0	0	0	$2k$	
$I_{2k+1}^{ns(012)}$	$-$	$2k+1$	$k+1$	0	$k+1$	0	0	0	0	$2k+1$	
$I_{2k}^{ns(012)}$	$Sp(k)$	$2k$	k	0	k	0	0	0	0	$2k$	
$I_{2k+1}^{*s(0\|1\|\|2)}$	$SO(4k+10)$	$k+2$	$k+2$	$k+1$	1	1	0	1	0	$2k+7$	
$I_{2k}^{*s(0\|1\|\|2)}$	$SO(4k+8)$	$k+2$	$k+1$	$k+1$	1	1	0	1	0	$2k+6$	
$I_{2k+1}^{*s(01\|\|2)}$	$SO(4k+10)$	$k+3$	$k+2$	$k+2$	1	1	0	0	0	$2k+7$	
$I_{2k}^{*s(01\|\|2)}$	$SO(4k+8)$	$k+2$	$k+2$	$k+1$	1	1	0	0	0	$2k+6$	
$I_{2k+1}^{*ns(01\|2)}$	$SO(4k+10)$	$2k+3$	$k+2$	1	$k+2$	1	0	0	0	$2k+7$	
$I_{2k}^{*ns(01\|2)}$	$SO(4k+8)$	$2k+2$	$k+2$	1	$k+1$	1	0	0	0	$2k+6$	
$IV^{*(0\|1\|2)}$	E_6	2	2	1	2	1	0	1	0	8	
$IV^{*(01\|2)}$	E_6	3	2	2	2	1	0	0	0	8	
$III^{*(01\|2)}$	E_7	3	3	2	2	1	0	0	0	9	

Table 3.4: Tate-like Forms for Singular Fibers. We present Tate-like forms for canonical fibers, listing the vanishing orders of the sections and of the discriminant, the resulting gauge group, and if necessary the conditions of validity for the forms. I_n fibers have four Tate forms depending on section separation and parity of n, while I_m^* have six forms again depending on where the sections intersect the fiber components and on parity of m. Exceptional singular fibers that could be realized canonically are also included.

to these symmetries. One finds that the following sets of vanishing orders give rise to fibrations which have codimension one singular fibers that are related by a relabelling of the coefficients of (3.1.1)

$$(n_1, n_2, n_3, n_5, n_6, n_7, n_8, n_9) \leftrightarrow (n_1, n_5, n_8, n_2, n_6, n_9, n_3, n_7)$$
$$(n_1+2, n_2+1, n_3, n_5+1, n_6, n_7, n_8, n_9) \leftrightarrow (n_1, n_2, n_3, n_5, n_6, n_7+1, n_8, n_9+1)$$
$$(n_1+1, n_2+1, n_3+1, n_5, n_6, n_7, n_8, n_9) \leftrightarrow (n_1, n_2, n_3, n_5, n_6, n_7, n_8+1, n_9+1) \tag{3.2.5}$$

and any composition thereof. In the analysis of Tate's algorithm for the quartic equation in $\mathbb{P}^{(1,1,2)}$ ([22]) these kind of symmetries were called lops. The first of these relations will be referred to as the $\mathbb{Z}_2$ symmetry, and the second and third relations, respectively, will be called lop

one and lop two.

These lop relations and the $\mathbb{Z}_2$ symmetry generate a family of equivalences by applying them repeatedly and in different orders. To choose an appropriate element of each equivalence class the procedure shall be as follows:

- Use the $\mathbb{Z}_2$ symmetry to fix $n_9 \geq n_7$.
- Apply lop one to reduce n_7 to 0.
- Apply lop two to reduce the least valued of n_8 and $n_9 - n_7$ to zero.
- Apply the $\mathbb{Z}_2$ symmetry.

In this way one can often choose a representative of a particular lop-equivalence class where $n_7 = n_9 = 0$. In the application of Tate's algorithm enhancements which move a form out of this lop-equivalence class will not be considered. In this way the redundancies inherent in the cubic equation (3.1.1) shall be removed. The remainder of this subsection shall be devoted to showing that these relations hold.

There is a $\mathbb{Z}_2$ symmetry that comes from the interchange

$$(n_1, n_2, n_3, n_5, n_6, n_7, n_8, n_9) \leftrightarrow (n_1, n_5, n_8, n_2, n_6, n_9, n_3, n_7)\,. \tag{3.2.6}$$

One can see this by observing that the equations for each form,

$$\mathfrak{s}_{1,n_1} w^3 + \mathfrak{s}_{2,n_2} w^2 x + \mathfrak{s}_{3,n_3} w x^2 + \mathfrak{s}_{5,n_5} w^2 y + \mathfrak{s}_{6,n_6} wxy + \mathfrak{s}_{7,n_7} x^2 y + \mathfrak{s}_{8,n_8} wy^2 + \mathfrak{s}_{9,n_9} xy^2 = 0\,, \tag{3.2.7}$$

and

$$\mathfrak{s}_{1,n_1} w^3 + \mathfrak{s}_{2,n_5} w^2 x + \mathfrak{s}_{3,n_8} w x^2 + \mathfrak{s}_{5,n_2} w^2 y + \mathfrak{s}_{6,n_6} wxy + \mathfrak{s}_{7,n_9} x^2 y + \mathfrak{s}_{8,n_3} wy^2 + \mathfrak{s}_{9,n_7} xy^2 = 0\,, \tag{3.2.8}$$

have identical vanishing orders up to the redefinition $x \leftrightarrow y$. This symmetry can be removed by only considering forms where, in order of preference,

$$\begin{aligned} n_7 &\geq n_9 \\ n_3 &\geq n_8 \\ n_2 &\geq n_5 \,. \end{aligned} \tag{3.2.9}$$

Furthermore there are symmetries that can occur in the partially resolved forms. One such, which was referred to as lop one above, is an equivalence between the vanishing orders

$$(n_1+2, n_2+1, n_3, n_5+1, n_6, n_7, n_8, n_9) \leftrightarrow (n_1, n_2, n_3, n_5, n_6, n_7+1, n_8, n_9+1)\,. \tag{3.2.10}$$

To see this consider first the geometry of the LHS after resolving the singularity at the point $x = y = z_1 = 0$ by the blow up $(x, y, z_1; \zeta_1)$[3]. It is clear that one can always do such a blow up as the n_i are, by definition, non-negative. The partially resolved geometry is

$$\begin{aligned} \mathfrak{s}_{1,n_1+2} w^3 z_1^{n_1+2} \zeta_1^{n_1} + \mathfrak{s}_{2,n_2+1} w^2 x z_1^{n_2+1} \zeta_1^{n_2} + \mathfrak{s}_{3,n_3} w x^2 z_1^{n_3} \zeta_1^{n_3} + \mathfrak{s}_{5,n_5+1} w^2 y z_1^{n_5+1} \zeta_1^{n_5} \\ + \mathfrak{s}_{6,n_6} wxy z_1^{n_6} \zeta_1^{n_6} + s_{7,n_7} x^2 y z_1^{n_7} \zeta_1^{n_7+1} + \mathfrak{s}_{7,n_7+1} x^2 y z_1^{n_7+1} \zeta_1^{n_7+2} \\ + \mathfrak{s}_{8,n_8} w y^2 z_1^{n_8} + s_{9,n_9} x y^2 z_1^{n_9} \zeta_1^{n_9+1} + \mathfrak{s}_{9,n_9+1} x y^2 z_1^{n_9+1} \zeta_1^{n_9+2} = 0\,, \end{aligned} \tag{3.2.11}$$

with the Stanley-Reiser ideal

$$\{wxy, w\zeta_1, xyz_1\}\,. \tag{3.2.12}$$

Similarly one can consider the RHS geometry after performing the small resolution $(w, z_2; \zeta_2)$ to separate the reducible divisor z_2. The geometry is

$$\begin{aligned} s_{1,n_1} w^3 z_2^{n_1} \zeta_2^{n_1+2} + s_{1,n_1+1} w^3 z_2^{n_1+1} \zeta_2^{n_1+3} + \mathfrak{s}_{1,n_1+2} w^3 z_2^{n_1+2} \zeta_2^{n_1+4} + s_{2,n_2} w^2 x z_2^{n_2} \zeta_2^{n_2+1} \\ + \mathfrak{s}_{2,n_2+1} w^2 x z_2^{n_2+1} \zeta_2^{n_2+2} + \mathfrak{s}_{3,n_3} w x^2 z_2^{n_3} \zeta_2^{n_3} + s_{5,n_5} w^2 y z_2^{n_5} \zeta_2^{n_5+1} + \mathfrak{s}_{5,n_5+1} w^2 y z_2^{n_5+1} \zeta_2^{n_5+2} \\ + \mathfrak{s}_{6,n_6} wxy z_2^{n_6} \zeta_2^{n_6} + \mathfrak{s}_{7,n_7+1} x^2 y z_2^{n_7+1} \zeta_2^{n_7} + \mathfrak{s}_{8,n_8} w y^2 z_2^{n_8} + \mathfrak{s}_{9,n_9+1} x y^2 z_2^{n_9+1} \zeta_2^{n_9} = 0\,, \end{aligned} \tag{3.2.13}$$

with Stanley-Reiser ideal

$$\{wxy, wz_2, xy\zeta_2\}\,. \tag{3.2.14}$$

Under the identification $z_1 \leftrightarrow \zeta_2$ and $\zeta_1 \leftrightarrow z_2$ it is observed that these equations and SR ideals are equivalent. Any multiplicity arising from this redundancy in (3.1.1) can

[3]The notation of ([74]) is used to spectify blow ups throughout this chapter.

be removed by combining it with one of the earlier constraints from the $\mathbb{Z}_2$ symmetry (3.2.9), $n_7 \geq n_9$, so as to choose to consider only forms which have $n_9 = 0$.

There is another relation among the partially resolved geometries, which was referred to as lop two,

$$(n_1+1, n_2+1, n_3+1, n_5, n_6, n_7, n_8, n_9) \leftrightarrow (n_1, n_2, n_3, n_5, n_6, n_7, n_8+1, n_9+1)\,. \quad (3.2.15)$$

Again this is seen by studying the partially resolved geometry explicitly. If $(n_1+1, n_2+1, n_3+1, n_5, n_6, n_7, n_8, n_9)$ is resolved by the small resolution $(y, z_1; \zeta_1)$ the blown up geometry is given by the equation

$$\begin{aligned}
&s_{1,n_1+1} w^3 z_1^{n_1+1}\zeta_1^{n_1} + \mathfrak{s}_{1,n_1+2} w^3 z_1^{n_1+2}\zeta_2^{n_1+1} + s_{2,n_2+1} w^2 x z_1^{n_2+1}\zeta_1^{n_2} + \mathfrak{s}_{2,n_2+2} w^2 x z_1^{n_2+2}\zeta_1^{n_2+1} \\
&+ s_{3,n_3+1} w x^2 z_1^{n_3+1}\zeta_1^{n_3} + \mathfrak{s}_{3,n_3+2} w x^2 z_1^{n_3+2}\zeta_1^{n_3+1} + \mathfrak{s}_{5,n_5} w^2 y z_1^{n_5}\zeta_1^{n_5} + \mathfrak{s}_{6,n_6} w x y z_1^{n_6}\zeta_1^{n_6} \\
&+ \mathfrak{s}_{7,n_7} x^2 y z_1^{n_7}\zeta_1^{n_7} + \mathfrak{s}_{8,n_8} w y^2 z_1^{n_8}\zeta_1^{n_8+1} + \mathfrak{s}_{9,n_9} x y^2 z_1^{n_9}\zeta_1^{n_9+1} = 0\,,
\end{aligned} \quad (3.2.16)$$

with SR-ideal

$$\{wxy, yz_1, wx\zeta_1\}\,. \quad (3.2.17)$$

On the other side if $(n_1, n_2, n_3, n_5, n_6, n_7, n_8+1, n_9+1)$ is resolved by the resolution $(w, x, z_2; \zeta_2)$ the geometry is then given as the vanishing of the hypersurface polynomial

$$\begin{aligned}
&\mathfrak{s}_{1,n_1} w^3 z_2^{n_1}\zeta_2^{n_1+1} + \mathfrak{s}_{2,n_2} w^2 x z_2^{n_2}\zeta_2^{n_2+1} + \mathfrak{s}_{3,n_3} w x^2 z_2^{n_3}\zeta_2^{n_3+1} + \mathfrak{s}_{5,n_5} w^2 y z_2^{n_5}\zeta_2^{n_5} \\
&+ \mathfrak{s}_{6,n_6} w x y z_2^{n_6}\zeta_2^{n_6} + \mathfrak{s}_{7,n_7} x^2 y z_2^{n_7}\zeta_2^{n_7} r + s_{8,n_8+1} w y^2 z_2^{n_8+1}\zeta_2^{n_8} + \mathfrak{s}_{8,n_8+2} w y^2 z_2^{n_8+2}\zeta_2^{n_8+1} \\
&+ s_{9,n_9+1} x y^2 z_2^{n_9+1}\zeta_2^{n_9} + \mathfrak{s}_{9,n_9+2} x y^2 z_2^{n_9+2}\zeta_2^{n_9+1} = 0\,,
\end{aligned} \quad (3.2.18)$$

with SR-ideal

$$\{wxy, wxz_2, \zeta_2 y\}\,. \quad (3.2.19)$$

These two geometries describe the same partially resolved space, and can be related by the interchange

$$z_1 \leftrightarrow \zeta_2\,, \quad \zeta_1 \leftrightarrow z_2\,. \quad (3.2.20)$$

3.2.3 Resolutions, Intersections, and the Shioda Map

To determine the Kodaira type, including the distribution of the marked points, of the codimension one singularity in the fibration specified by (3.1.1) one often explicitly constructs the resolved geometry via a sequence of algebraic resolutions. In the context of elliptic fibrations such resolutions have been constructed in ([18,48,65,68,69,74–78]). In this section we set up the framework to discuss the resolved geometries and the intersection computations, for example of $U(1)$ charges of matter curves, that are carried out as part of the analysis of the singular fibers found. In particular details are given about the embedding of the fibration as a hypersurface in an ambient fivefold, the details of how the intersection numbers between curves and fibral divisors are computed, and on the construction of the $U(1)$ charge generators.

Consider the ambient fivefold $X_5 = \mathbb{P}^2(\mathcal{O}\oplus\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta))$ which is the projectivization of line bundles over a base space B_3. The elliptically fibered Calabi-Yau fourfold will be realized as the hypersurface in this X_5 cut out by the cubic equation (3.1.1). The terms in the homogeneous polynomial are then sections of the following line bundles

Section	Bundle
w	$\mathcal{O}(\sigma)$
x	$\mathcal{O}(\sigma+\alpha)$
y	$\mathcal{O}(\sigma+\beta)$
z	$\mathcal{O}(S_G)$
$\mathfrak{s}_{1,j}$	$\mathcal{O}(c_1+\alpha+\beta-jS_G)$
$\mathfrak{s}_{2,j}$	$\mathcal{O}(c_1+\beta-jS_G)$
$\mathfrak{s}_{3,j}$	$\mathcal{O}(c_1-\alpha+\beta-jS_G)$
$\mathfrak{s}_{5,j}$	$\mathcal{O}(c_1+\alpha-jS_G)$
$\mathfrak{s}_{6,j}$	$\mathcal{O}(c_1-jS_G)$
$\mathfrak{s}_{7,j}$	$\mathcal{O}(c_1-\alpha-jS_G)$
$\mathfrak{s}_{8,j}$	$\mathcal{O}(c_1+\alpha-\beta-jS_G)$
$\mathfrak{s}_{9,j}$	$\mathcal{O}(c_1-\beta-jS_G)$

(3.2.21)

Here c_1 is a shorthand notation for $\pi^*c_1(B_3)$. In practice, the first step in any explicit determination of a singular fiber is to blow up the $\mathbb{P}^2$ fibration to a dP_2 fibration by the substitution $w \to l_1 l_2 w$, $x \to l_1 x$, and $y \to l_2 y$ and taking the proper transform, as was also the procedure in ([1, 24–26]).

The geometry is then specified by the equation

$$\mathfrak{s}_1 l_1^2 l_2^2 w^3 + \mathfrak{s}_2 l_1^2 l_2 w^2 x + \mathfrak{s}_3 l_1^2 w x^2 + \mathfrak{s}_5 l_1 l_2^2 w^2 y + \mathfrak{s}_6 l_1 l_2 wxy + \mathfrak{s}_7 l_1 x^2 y + \mathfrak{s}_8 l_2^2 w y^2 + \mathfrak{s}_9 l_2 x y^2 = 0\,, \tag{3.2.22}$$

in dP_2. After these blow ups the fiber coordinates in this equation are sections of the line bundles

$$
\begin{array}{c|c}
\text{Section} & \text{Bundle} \\
\hline
w & \mathcal{O}(\sigma - F_1 - F_2) \\
x & \mathcal{O}(\sigma + \alpha - F_1) \\
y & \mathcal{O}(\sigma + \beta - F_2) \\
l_1 & \mathcal{O}(F_1) \\
l_2 & \mathcal{O}(F_2)
\end{array}
\tag{3.2.23}
$$

As can be seen from the blow ups which mapped $\mathbb{P}^2$ to dP_2 the marked point $[0:0:1]$ has been mapped to the exceptional divisor l_1, similarly for $[0:1:0]$ and l_2. As such the marked points Σ_0, Σ_1, and Σ_2 have been related to the divisors l_1, w, and l_2 respectively.

As the marked points form sections they are restricted to intersect, in codimension one, only a single multiplicity one component of the singular fiber ([79]).

The dP_2 intersection ring is not freely generated due to the projective relations which hold in dP_2. These relations are, using standard projective coordinate notation,

$$[wl_1l_2 : xl_1 : yl_2]\,, \quad [w:x]\,, \quad [w:y]\,. \tag{3.2.24}$$

These correspond to the relations in the intersection ring

$$
\begin{aligned}
\sigma \cdot (\sigma + \alpha) \cdot (\sigma + \beta) &= 0 \\
(\sigma - F_1 - F_2) \cdot (\sigma - F_1) &= 0 \\
(\sigma - F_1 - F_2) \cdot (\sigma - F_2) &= 0\,.
\end{aligned}
\tag{3.2.25}
$$

The strategy, as it was in ([74, 77]), will be to choose a basis of the intersection ring and repeatedly apply these relations, including any that come from exceptional divisor classes introduced in the resolution. In this way the intersection numbers between curves and fibral divisors can be computed. In this chapter the resolutions and intersections were carried out using the Mathematica package `Smooth` ([80]).

Given an elliptic fibration with multiple rational sections there remains the construction of the generators of the $U(1)$ symmetries, that is the generators of the Mordell-Weil group. The Mordell-Weil group is a finitely generated abelian group ([81])

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathcal{G}\,, \tag{3.2.26}$$

where $\mathcal{G}$ is some finite torsion group[4]. There is a map, known as the Shioda map, which constructs from rational sections the generators of the Mordell-Weil group. This map is discussed in detail in ([23, 84, 85]).

[4]We shall not concern ourselves with $\mathcal{G}$ in this chapter, but some investigations are ([82, 83]).

The Shioda map associates to each rational section, σ_i, a divisor $s(\sigma_i)$ such that

$$\begin{aligned} s(\sigma_i) \cdot F_j &= 0 \\ s(\sigma_i) \cdot B &= 0 \,, \end{aligned} \tag{3.2.27}$$

where F_j are the exceptional curves and B is the dual to the class of the base B_3. Reduction on the F_j gives rise to gauge bosons which should be uncharged under the abelian gauge symmetry. This is ensured by the conditions (3.2.27).

The charge of a particular matter curve C with respect to the $U(1)$ generator associated to the rational section σ_i is given by the intersection number $s(\sigma_i) \cdot C$. The constraints (3.2.27) determine the $U(1)$ charges from $s(\sigma_i)$ up to an overall scale. We shall always consider the zero-section to be the rational section associated with the introduction of the l_1 in the blow up to dP_2.

As was alluded to in section 3.2.1 it is not always the case that a fibration that arises from the algorithm can be specified purely in terms of the vanishing orders of the coefficients. Sometimes it is necessary to also include some specialization of the coefficients in the z-expansion of the coefficients of the equation. Consider a discriminant of the form

$$\Delta = (AB - CD) z^n + \mathcal{O}(z^{n+1}) \,. \tag{3.2.28}$$

An enhancement that would enhance this singularity would be where $AB - CD = 0$. The solution of this polynomial cannot in general be specified in terms of the vanishing order of A, B, C, and D. In appendix A.1 we collect the solutions to several polynomials of this form which come up repeatedly in the application of Tate's algorithm to (3.1.1). The solution to this particular polynomial is

$$\begin{aligned} A &= \sigma_1 \sigma_2 \\ B &= \sigma_3 \sigma_4 \\ C &= \sigma_1 \sigma_3 \\ D &= \sigma_2 \sigma_4 \,, \end{aligned} \tag{3.2.29}$$

where the pairs (σ_1, σ_4) and (σ_2, σ_3) are coprime. It is not generally possible to perform some shift of the coordinates in (3.1.1) to return this solution to an expression involving just vanishing orders. This is notably different from Tate's algorithm as carried out on the Weierstrass equation in ([21]); there the equation includes monic terms unaccompanied by any coefficient, which often allows one to shift the variables to absorb these non-canonical like solutions into higher vanishing orders of the model.

3.3 Tate's Algorithm

In this section we will proceed through the algorithm ([20, 21]), considering the discriminant of the elliptic fibration order by order in the expansion in terms of the base coordinate z. By enhancing the fiber of our elliptic fibration, we will see under which conditions on the sections $\mathfrak{s}_i$ the order of the discriminant will enhance and then study the resulting singular fibers. This will be done systematically up to singular I_5 fibers for phenomenological reasons and in section 3.5 we will provide details for some of the exceptional singular fibers. In a step-by-step application of Tate's algorithm to the elliptic fibration (3.1.1) we find the various different types of Kodaira singular fibers decorated with the information of which sections intersect which components. The discriminant reflects the different ways in which the sections can intersect the multiplicity one fiber components (as explained in section 3.2.3), thus giving rise to an increased number of singular fibers over fibrations with fewer rational sections. The analysis will be carried out in parallel both for canonical models (determined only by the vanishing orders of the sections) and for non-canonical models (which require additional specialization arising from solving polynomials in the discriminant.)

3.3.1 Starting Points

In the following we will assume that the fibration develops a singularity along the locus $z = 0$ in the base. A singularity can be characterized by one of the following two criteria:

- The leading order of the discriminant as a series expansion in z must vanish.
- The derivatives of $\mathfrak{D}|_{z=0}$ in an affine patch must vanish along the $z = 0$ locus, where $\mathfrak{D}$ is the equation for the fibration.

Since the leading order of the discriminant is a complicated and unenlightening expression, we will not present it here and instead study the derivatives of the equation of the fibration. This will turn out to be significantly simpler and we will see that the discriminant will enhance upon substitution of the conditions found by the derivative analysis. On the other hand, throughout our study of higher order singularities we will look only at the discriminant ignoring the derivative approach.

Let us then study the equation for the elliptic fibration in the affine patch with coordinates (x, y), that is, where we can scale such that $w = 1$. Along the locus $z = 0$ we assume that the fiber becomes singular at the point (x_0, y_0) and require the derivatives to vanish

$$\begin{aligned}\mathfrak{D}|_{z=0} =& s_{1,0} + s_{2,0}x_0 + s_{3,0}x_0^2 + s_{5,0}y_0 + s_{6,0}x_0y_0 + s_{7,0}x_0^2y_0 + s_{8,0}y_0^2 + s_{9,0}x_0y_0^2 = 0 \\ \partial_x\mathfrak{D}|_{z=0} =& s_{2.0} + 2s_{3,0}x_0 + s_{6,0}y_0 + 2s_{7,0}x_0y_0 + s_{9,0}y_0^2 = 0 \\ \partial_y\mathfrak{D}|_{z=0} =& s_{5,0} + s_{6,0}x_0 + s_{7,0}x_0^2 + 2s_{8,0}y_0 + 2s_{9,0}x_0y_0 = 0\,.\end{aligned} \tag{3.3.1}$$

We can solve for $s_{2,0}$ and $s_{5,0}$ from the last two equations

$$\begin{aligned}s_{2,0} =& -2s_{3,0}x_0 - y_0(s_{6,0} + 2s_{7,0}x_0 + s_{9,0}y_0) \\ s_{5,0} =& -x_0(s_{6,0} + s_{7,0}x_0) - 2(s_{8,0} + s_{9,0}x_0)y_0\,.\end{aligned} \tag{3.3.2}$$

Upon substitution in the first equation we can solve for $s_{1,0}$

$$s_{1,0} = s_{3,0}x_0^2 + y_0(s_{8,0}y_0 + x_0(s_{6,0} + 2s_{7,0}x_0 + 2s_{9,0}y_0))\,. \tag{3.3.3}$$

When $s_{1,0}, s_{2,0}$ and $s_{5,0}$ satisfy the above requirements the discriminant indeed enhances to first order. We can bring the equation of the fibration in a canonical form, depending only on the vanishing orders of the coefficients, by performing the following coordinate shift

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x - x_0w \\ y - y_0w \end{pmatrix}\,. \tag{3.3.4}$$

We see that the singularity now sits at the origin of the affine patch and has generic coefficients in addition to $\{s_{1,0} = s_{2,0} = s_{5,0} = 0\}$. This is an I_1 singular fiber, which is the only fiber at vanishing order $\mathrm{ord}_z(\Delta) = 1$ in Kodaira's classification. That this is indeed an I_1 fiber can also be seen by performing a linear approximation around the singular point and noting that we obtain two distinct tangent lines, which shows that this is indeed an ordinary double point. Since there is only one fiber component, all the three sections will intersect it, and we will denote the singular fiber

$$I_1^{(012)}: \quad (1,1,0,1,0,0,0,0)\,. \tag{3.3.5}$$

This does not exhaust the possible ways to solve the three equations in (3.3.1). Indeed, we can look at the affine subspace $y = 0$ and see that we can find additional solutions. Note that we will not consider here the case $x = 0$ as this is related by the $\mathbb{Z}_2$ symmetry discussed in section 3.2.2. The partial derivatives now read

$$\begin{aligned}\mathfrak{D}|_{z=y=0} =& s_{1,0} + s_{2,0}x_0 + s_{3,0}x_0^2 = 0 \\ \partial_x\mathfrak{D}|_{z=y=0} =& s_{2,0} + 2s_{3,0}x_0 = 0 \\ \partial_y\mathfrak{D}|_{z=y=0} =& s_{5,0} + s_{6,0}x_0 + s_{7,0}x_0^2 = 0\,.\end{aligned} \tag{3.3.6}$$

We see that if we require $\{s_{1,0} = s_{2,0} = s_{3,0} = 0\}$ the three equations are satisfied for the two solutions of the quadratic equation $\{s_{5,0} + s_{6,0}x_0 + s_{7,0}x_0^2 = 0\}$, which are the two singular points of an I_2 Kodaira fiber as the discriminant enhances to vanishing order $\Delta(z^2)$. Indeed, looking at the equation of the fiber, we see that this splits in two components

$$\begin{aligned} D_1: &\qquad z = y = 0 \\ D_2: &\qquad z = s_{5,0}w^2 + s_{6,0}wx + s_{7,0}x^2 + (s_{8,0}w + s_{9,0}x)y = 0\,. \end{aligned} \tag{3.3.7}$$

The two components indeed intersect in two different points, thus showing that this is an I_2 singular fiber. One of the sections intersects one component, while the two remaining sections intersect the other, so we will denote this fiber as

$$I_2^{(01|2)}: \quad (1,1,1,0,0,0,0,0)\,. \tag{3.3.8}$$

These two fibers represent the starting points for the analysis to be carried out in the remainder of this section. Given the equation for the fibration, we can ask whether z divides any of the coefficients $\mathfrak{s}_i$. Then we can conclude, inside our preferred lop-equivalence class, the following:

- If $z \nmid \mathfrak{s}_1$ and $z \nmid \mathfrak{s}_2$ then the fiber over the locus $\{z = 0\}$ is smooth.
- If $z \mid \mathfrak{s}_1, z \mid \mathfrak{s}_2$ and $z \mid \mathfrak{s}_3$ then we can carry on the analysis as in the next section and check whether the singularity is simply $I_2^{(01|2)}$ or some other enhanced kind.
- If $z \mid \mathfrak{s}_1, z \mid \mathfrak{s}_2$ and $z \mid \mathfrak{s}_5$ we will instead start our analysis from an $I_1^{(012)}$ singular fiber. It is important to notice that in this part of the algorithm we will not let $z \mid \mathfrak{s}_3$ as this case is covered in the previous branch.

3.3.2 Enhancements from $\mathrm{ord}_z(\Delta) = 1$

From the previous section we have found exactly one starting point for the algorithm which has a discriminant linear in z: $(1,1,0,1,0,0,0,0)$. In this section we shall study the various ways that this I_1 singular fiber can enhance. The discriminant of the $(1,1,0,1,0,0,0,0)$ fibration is

$$\Delta = s_{1,1}s_{3,0}s_{8,0}(s_{6,0}^2 - 4s_{3,0}s_{8,0})(s_{7,0}^2s_{8,0} - s_{6,0}s_{7,0}s_{9,0} + s_{3,0}s_{9,0}^2)z + \mathcal{O}(z^2)\,, \tag{3.3.9}$$

up to numerical factors. The discriminant factors into five distinct terms which will enhance the discriminant, and thus the singular fiber, when they vanish. As this set of vanishing orders is specifying a fibration where $z \nmid \mathfrak{s}_3$ then we cannot consider the

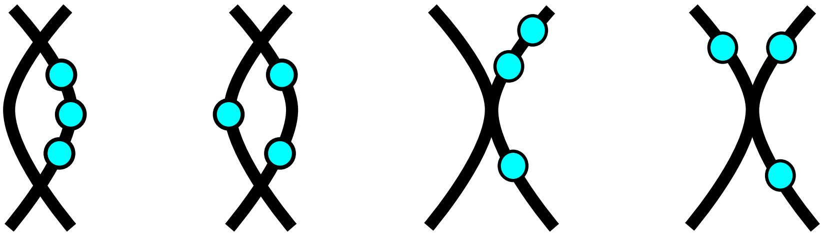


Figure 3.1: The type I_2 and type III singular fibers with the possible locations of the three marked points denoted by the blue nodes. Respectively these are $I_2^{(ijk)}$, $I_2^{(i|jk)}$, $II^{(ijk)}$ and $II^{(i|jk)}$ fibers.

situation where $s_{3,0} = 0$. Equivalently, because of the $\mathbb{Z}_2$ symmetry explained in section 3.2.2, we cannot consider $s_{8,0}$ vanishing.

First let us consider the simple case where $s_{1,1} = 0$, which is equivalent to stating that $z^2 \mid \mathfrak{s}_1$. Then $z^2 \mid \Delta$ and the singular fiber type, determined by resolving the singularity explicitly as explained in section 3.2.3, is I_2. The three rational sections all intersect one of the two components of the I_2 fiber

$$I_2^{(012)} \,:\, (2,1,0,1,0,0,0,0)\,, \tag{3.3.10}$$

listed in table 3.1.

The discriminant can also be enhanced in order by allowing z to divide either of the two polynomials in (3.3.9). Let us first consider the situation where $s_{6,0}^2 - 4s_{3,0}s_{8,0}$ vanishes. The solution to this equation over this unique factorization domain is given in appendix A.1 and states that

$$\begin{aligned} s_{6,0} &= \mu\sigma_3\sigma_8 \\ s_{3,0} &= \mu\sigma_3^2 \\ s_{8,0} &= \mu\sigma_8^2\,. \end{aligned} \tag{3.3.11}$$

The discriminant then enhances so that $z^2 \mid \Delta$. To determine the type of singular fiber here let us consider the equation of the single component of the I_1 fiber which is being enhanced

$$(s_{3,0}x^2 + s_{6,0}xy + s_{8,0}y^2) + xy(s_{7,0}x + s_{9,0}y) = 0\,. \tag{3.3.12}$$

If $s_{6,0}^2 - 4s_{3,0}s_{8,0} = 0$ then the quadratic part of the equation factors into a square which does not divide the cubic terms; this is exactly the form of the equation for a cusp, which is a type II fiber. Therefore we have observed the fiber

$$II_{nc}^{(012)}: \quad \left\{ \begin{array}{c} (1,1,0,1,0,0,0,0) \\ [-,-,\mu\sigma_3^2,-,2\mu\sigma_3\sigma_8,-,\mu\sigma_8^2,-] \end{array} \right\}, \tag{3.3.13}$$

from table 3.1.

Finally we can consider the singular fiber that occurs when the second polynomial in Δ vanishes: $s_{7,0}^2 s_{8,0} - s_{6,0}s_{7,0}s_{9,0} + s_{3,0}s_{9,0}^2 = 0$. Appendix A.1 lists four generic solutions of this polynomial, three canonical and one non-canonical, which are:

$$s_{7,0} = s_{9,0} = 0$$

$$s_{7,0} = s_{3,0} = 0$$

$$s_{8,0} = s_{9,0} = 0$$

$$s_{7,0} = \sigma_1\sigma_2\,, \quad s_{9,0} = \sigma_1\sigma_3\,, \quad s_{8,0} = \sigma_3\sigma_4\,, \quad s_{3,0} = \sigma_2\sigma_5\,, \quad s_{6,0} = \sigma_2\sigma_4 + \sigma_3\sigma_5\,. \tag{3.3.14}$$

Any of the three canonical solutions will remove us from our preferred lop-equivalence class and so we do not consider them as they will give rise to a redundancy of singular fiber types. The only solution to consider therefore is the non-canonical one. The fiber found at this locus is another I_2 fiber, which can be written as

$$I_{2,nc}: \quad \left\{ \begin{array}{c} (1,1,0,1,0,0,0,0) \\ [-,-,\sigma_2\sigma_5,-,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_1\sigma_2,\sigma_3\sigma_4,\sigma_1\sigma_3] \end{array} \right\}, \tag{3.3.15}$$

Table 3.1 is then complete up to second order, once we also include the $I_2^{(01|2)}$ which was found in the previous section as one of the alternate starting points in the $z \mid \mathfrak{s}_3$ branch.

3.3.3 Enhancements from $\mathrm{ord}_z(\Delta) = 2$

We will now consider the enhancement of the four previously found fibrations which have a discriminant with vanishing order two in z. In this section we shall include the details only of those enhancements that have some non-standard behaviour.

The fibrations $(2,1,0,1,0,0,0,0)$ and $(1,1,1,0,0,0,0,0)$ can contain, respectively, in their discriminants polynomials with five and seven terms. These are not polynomials that are discussed in appendix A.1 as their solutions are not known in full generality. In lieu of a complete solution we consider non-generic but canonical type solutions which allow us to obtain singular fibers of a particular type which would be unobtainable without determining a full, generic solution to these polynomials.

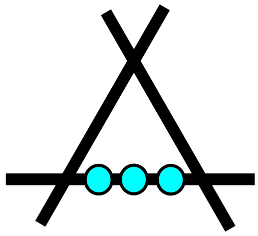


Figure 3.2: The type I_3 singular fibers with the locations of the three marked points denoted by the blue nodes. Respectively these are $I_3^{(ijk)}$, $I_3^{(ij|k)}$ and $I_3^{(i|j|k)}$ fibers.

3.3.4 Polynomial enhancement in the $z \nmid \mathfrak{s}_3$ branch

The discriminant of the equation for the $(2,1,0,1,0,0,0,0)$ singular fiber contains the polynomial

$$P = s_{8,0}s_{2,1}^2 - s_{5,1}s_{6,0}s_{2,1} + s_{1,2}s_{6,0}^2 + s_{3,0}\left(s_{5,1}^2 - 4s_{1,2}s_{8,0}\right) . \tag{3.3.16}$$

As the most general solution for this five-term polynomial is not known we propose here two specific solutions. The first is a canonical solution obtained by setting $s_{1,2} = s_{2,1} = s_{5,1} = 0$. As a consequence $z^3 \mid \Delta$ and we find an $I_3^{ns(012)}$ singular fiber

$$I_3^{ns(012)} \, : \, (3,2,0,2,0,0,0,0) . \tag{3.3.17}$$

Recalling the split/non-split monodromy distinction in Tate's algorithm, we see only two components in this singular fiber. One of the fiber curves decomposes when the component of the discriminant, $s_{6,0}^2 - 4s_{3,0}s_{8,0}$ has the form of a perfect, non-zero square.

The second non-general solution to the five-term polynomial we consider here is found by canonically setting $s_{1,2} = 0$, and then the five term polynomial reduces to

$$P|_{(s_{1,2}=0)} = s_{2,1}^2 s_{8,0} - s_{2,1}s_{6,0}s_{5,1} + s_{5,1}^2 s_{3,0} . \tag{3.3.18}$$

We notice that we cannot set $s_{3,0}$ to zero because we are in the $z \nmid \mathfrak{s}_3$ part of the algorithm (and by $\mathbb{Z}_2$ symmetry we cannot set to zero $s_{8,0}$ either). Moreover we just considered the canonical solution given by setting $s_{2,1} = s_{5,1} = 0$. We are then left with imposing the non-canonical solution given in appendix A.1

$$s_{2,1} = \sigma_1\sigma_2 \, , \quad s_{5,1} = \sigma_1\sigma_3 \, , \quad s_{8,0} = \sigma_3\sigma_4 \, , \quad s_{3,0} = \sigma_2\sigma_5 \, , \quad s_{6,0} = \sigma_2\sigma_4 + \sigma_3\sigma_5 \, . \tag{3.3.19}$$

The resulting singular fiber is then an $I_{3,nc}^{s(012)}$

$$I_{3,nc}^{s(012)} : \quad \left\{ \begin{array}{c} (3,1,0,1,0,0,0,0) \\ {[-,\sigma_1\sigma_2,\sigma_2\sigma_5,\sigma_1\sigma_3,\sigma_2\sigma_4+\sigma_3\sigma_5,-,\sigma_3\sigma_4,-]} \end{array} \right\} , \tag{3.3.20}$$

3.3.5 Polynomial enhancement in the $z \mid \mathfrak{s}_3$ branch

The other relevant details we will provide concern enhancements from the singular $I_2^{(01|2)}$ which has vanishing orders $(1,1,1,0,0,0,0,0)$. The discriminant contains a seven-term polynomial

$$\begin{aligned} P = s_{3,1}^2 s_{5,0}^2 + s_{7,0}(s_{2,1}^2 s_{5,0} - s_{1,1}s_{2,1}s_{6,0} + s_{1,1}^2 s_{7,0}) + \\ + s_{3,1}(-s_{2,1}s_{5,0}s_{6,0} + s_{1,1}(s_{6,0}^2 - 2s_{5,0}s_{7,0}))\,. \end{aligned} \tag{3.3.21}$$

Since a generic solution is not known for this polynomial, we again take advantage of a simple canonical solution given by $s_{1,1} = s_{2,1} = s_{3,1} = 0$. We see that $z^4 \mid \Delta$ and we observe a singular $I_4^{ns(01||2)}$

$$I_4^{ns(01||2)} : (2,2,2,0,0,0,0,0)\,. \tag{3.3.22}$$

As in the previous case, we notice that the component of the discriminant $s_{6,0}^2 - 4s_{5,0}s_{7,0}$ provides the condition for the split/non-split distinction. If this quantity is a perfect, non-zero square, then applying the solution given in appendix A.1 we have a split $I_{4,nc}^{s(01||2)}$

$$I_{4,nc}^{s(01||2)} : \left\{ \begin{array}{c} (2,2,2,0,0,0,0,0) \\ [-,-,-,\sigma_1\sigma_3, \sigma_1\sigma_2 + \sigma_3\sigma_4, \sigma_2\sigma_4, -, -] \end{array} \right\}, \tag{3.3.23}$$

As in the previous subsection, we notice that if we only require $s_{1,1} = 0$ the seven term polynomial reduces to the usual three-term one

$$P|_{(s_{1,2}=0)} = s_{5,0}(s_{3,1}^2 s_{5,0} - s_{3,1}s_{6,0}s_{2,1} + s_{2,1}^2 s_{7,0})\,. \tag{3.3.24}$$

The solution involving setting $s_{5,0}$ to zero in addition to $s_{1,2}$ would give the fibration defined by the vanishing orders $(2,1,1,1,0,0,0,0)$ which is an $I_3^{s(01|2)}$ fiber

$$I_3^{s(01|2)} : (2,1,1,1,0,0,0,0)\,. \tag{3.3.25}$$

We can also apply the non-canonical solution of appendix A.1 to the three-term component

$$s_{2,1} = \sigma_1\sigma_2\,, \quad s_{3,1} = \sigma_1\sigma_3\,, \quad s_{7,0} = \sigma_3\sigma_4\,, \quad s_{5,0} = \sigma_2\sigma_5\,, \quad s_{6,0} = \sigma_2\sigma_4 + \sigma_3\sigma_5\,. \tag{3.3.26}$$

Upon substitution we find an $I_{3,nc}^{s(01|2)}$ singular fiber

$$II_{nc}^{(012)} : \left\{ \begin{array}{c} (1,1,0,1,0,0,0,0) \\ [-,-,\mu\sigma_3^2, -, 2\mu\sigma_3\sigma_8, -, \mu\sigma_8^2, -] \end{array} \right\}, \tag{3.3.27}$$

$$I_{3,nc}^{s(01|2)} : (2,1,1,0,0,0,0,0)\,, [-,\sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_5, \sigma_2\sigma_4 + \sigma_3\sigma_5, \sigma_3\sigma_4, -, -]\,. \tag{3.3.28}$$

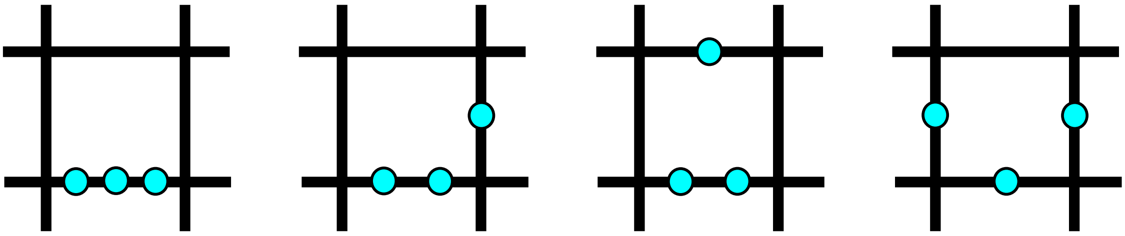


Figure 3.3: The I_4 singular fibers and the decorations detailing where the rational sections can intersect. The fibers shown are $I_4^{(ijk)}$, $I_4^{(ij|k)}$, $I_4^{(ij||k)}$ and $I_4^{(i|j|k)}$ fibers.

3.3.6 Enhancements from $\mathrm{ord}_z(\Delta) = 3$

We now proceed to consider enhancements of the discriminant starting from the fibers with $\mathrm{ord}_z(\Delta) = 3$, listed in table 3.1, and we report here the cases that deserve mention due to some peculiarity. In particular we will consider distinctions between split and non-split singular fibers and an instance where we will need to consider the structure of the algorithm in order not to reproduce singular fibers already obtained.

3.3.7 Split/non-split distinction

We recall that in the previous section we found an $I_3^{ns(012)}$ singular fiber and we now determine the enhancements of this fiber. The discriminant takes the form

$$\Delta = s_{1,3}s_{3,0}s_{8,0}(s_{6,0}^2 - 4s_{3,0}s_{8,0})(s_{7,0}^2 s_{8,0} - s_{6,0}s_{7,0}s_{9,0} + s_{3,0}s_{9,0}^2)z^3 + \mathcal{O}(z^4)\,. \qquad (3.3.29)$$

The simple enhancement $s_{1,3} = 0$ will produce an $I_4^{ns(012)}$ singular fiber

$$I_4^{ns(012)} \,:\, (4, 2, 0, 2, 0, 0, 0, 0)\,. \qquad (3.3.30)$$

As already observed, the discriminant component $s_{6,0}^2 - 4s_{3,0}s_{8,0}$ indicates that when this quantity is a perfect, non-zero square, we obtain the split version of the singular fiber. Applying the solution in appendix A.1 we then find the singular $I_{4,nc}^{s(012)}$

$$I_{4,nc}^{s(012)} \,:\, (4, 2, 0, 2, 0, 0, 0, 0)\,,\, [-, -, \sigma_1\sigma_3, -, \sigma_1\sigma_2 + \sigma_3\sigma_4, -, \sigma_2\sigma_4, -]\,. \qquad (3.3.31)$$

Another instance where the split/non-split distinction arises is in the case of type IV fibers. Consider the singular $III_{nc}^{(012)}$ listed in table 3.1. This has discriminant

$$\Delta = \mu^6\sigma_3\sigma_8(s_{5,1}\sigma_3 - s_{2,1}\sigma_8)(s_{9,0}\sigma_3 - s_{7,0}\sigma_8)z^3 + \mathcal{O}(z^4)\,. \qquad (3.3.32)$$

We remark that this was obtained in the algorithm by an application of the non-canonical solution to $s_{6,0}^2 - 4s_{3,0}s_{8,0} = 0$ and therefore σ_3 and σ_8 are coprime. Enhancing the

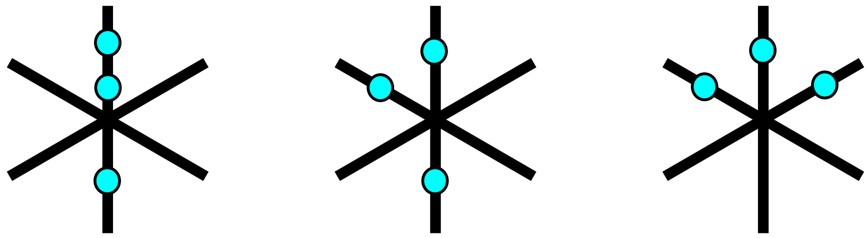


Figure 3.4: The IV fibers. We denote by the blue nodes the components of the fiber which are intersected by the sections. In the order, the fiber shown are $IV^{(ijk)}$, $IV^{(i|jk)}$ and $IV^{(i|j|k)}$ fibers.

discriminant by solving non-canonically the first of the two-term polynomials requires setting

$$s_{5,1} = \xi_1\xi_2, \qquad \sigma_3 = \xi_3, \qquad s_{2,1} = \xi_1\xi_3, \qquad \sigma_8 = \xi_2\,. \tag{3.3.33}$$

Where coprimality of (σ_3, σ_8) was used in order to set $\xi_4 = 1$. The singular fiber corresponding to this enhancement is a type $IV^{ns(012)}_{nc^2}$

$$IV^{ns(012)}_{nc^2}\ :\ (2,1,0,1,0,0,0,0)\,,\ [-, \xi_1\xi_3, \mu\xi_3^2, \xi_1\xi_2, 2\mu\xi_2\xi_3, -, \mu\xi_2^2, -]\,. \tag{3.3.34}$$

Then the discriminant indicates the quantity that needs to be a perfect square in order for the fiber to become a split type $IV^{s(012)}_{nc^3}$. This is $\xi_1^2 - 4\mu s_{1,2}$, and applying the solution in appendix A.1 we find

$$IV^{s(012)}_{nc^3}\ :\ (2,1,0,1,0,0,0,0)\,,\ [\delta_2\delta_4, \xi_3(\delta_1\delta_2+\delta_3\delta_4), \delta_1\delta_3\xi_3^2, \xi_2(\delta_1\delta_2+\delta_3\delta_4), 2\delta_1\delta_3\xi_2\xi_3, -, \delta_1\delta_3\xi_2^2, -]\,. \tag{3.3.35}$$

3.3.8 Commutative enhancement structure of the algorithm

We consider enhancements from the $III^{(1|02)}_{nc^2}$ fiber type. This was found by applying twice the solutions in appendix A.1. Schematically

$$I_1^{(012)} \longrightarrow I_{2,nc}^{(1|02)} \longrightarrow III^{(1|02)}_{nc^2}\,. \tag{3.3.36}$$

Noting that in the last step a coprimality condition had to be imposed, the discriminant of this singular fiber takes the form

$$\Delta = (s_{1,1}^3\xi_2^6\xi_3^6\xi_4^6)z^3 + \mathcal{O}(z^4)\,. \tag{3.3.37}$$

We see that requiring the vanishing of any of the ξ_i would imply setting to zero two among $s_{7,0}, s_{9,0}, s_{3,0}, s_{8,0}$, but we are not allowing the vanishing of any of the those sections to remain in our lop equivalence class or because we are in the $z \nmid \mathfrak{s}_3$ branch of

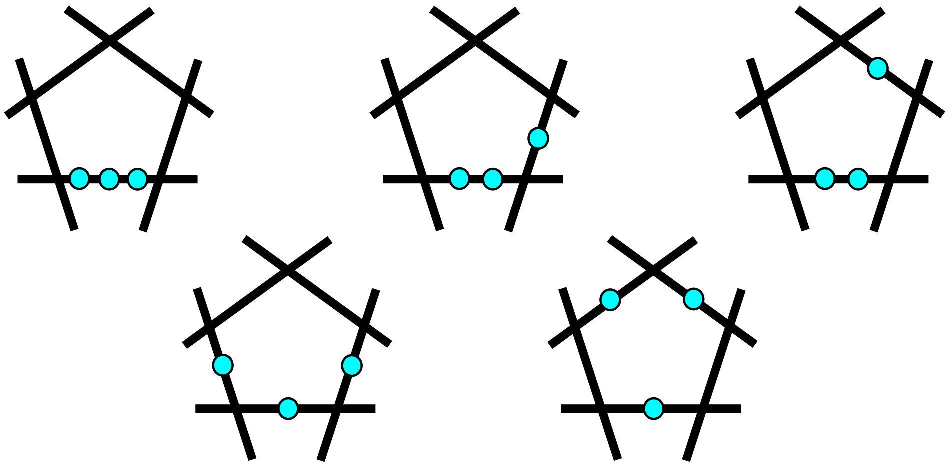


Figure 3.5: The I_5 singular fibers. The possible intersections of the sections with the singular fibers are denoted by the positions of the blue nodes. The fibers shown in the first row are $I_5^{(ijk)}$, $I_5^{(ij|k)}$ and $I_5^{(ij||k)}$, whereas the fibers shown in the second row are, respectively, $I_5^{(i|j|k)}$ and $I_5^{(i|j||k)}$.

the algorithm. Moreover, we have considered the case $s_{1,1} = 0$ in another part of the algorithm (specifically in going $I_1^{(012)} \to I_2^{(012)}$). We can therefore conclude that all the enhancements would just reproduce singular fibers found in other parts of the algorithm. The order in which the enhancements are carried out is of no importance, but it is crucial, in particular with non-canonical fibers, to keep track of which enhancements would reproduce fiber types already obtained.

3.3.9 Enhancements from $\mathrm{ord}_z(\Delta) = 4$

In this section we will proceed with the algorithm by again mentioning only enhancements which require comment. In particular we will deal with the structure of obstructions to full generality due to the complexity of polynomials in the discriminant, we will encounter the distinction between split and semi-split fibers for I_0^* and we will provide details for one of the I_{5,nc^3}, obtained by solving non-canonically polynomials in the discriminant three times.

3.3.10 Obstruction from polynomial enhancement

At vanishing order of the discriminant $\mathrm{ord}_z(\Delta) = 4$ we find again the two obstructions to full generality encountered at $\mathrm{ord}_z(\Delta) = 2$, i.e. the same five-term and seven-term polynomials. These come up respectively in the discriminant of the singular fibers $I_4^{ns(012)}$

and $I_4^{ns(01||2)}$, and will in fact be present at every even vanishing order in the discriminants of $I_{2n}^{ns(012)}$ and $I_{2n}^{ns(01|^n2)}$. We therefore review the singular fibers that we obtain from the enhancements. More details can be found in section 3.3.3.

The discriminant of the singular fiber $I_4^{ns(012)}$: $(4,2,0,2,0,0,0,0)$ contains a component

$$\Delta \supset P = s_{8,0}s_{2,2}^2 - s_{5,2}s_{6,0}s_{2,2} + s_{1,4}s_{6,0}^2 + s_{3,0}\left(s_{5,2}^2 - 4s_{1,4}s_{8,0}\right) . \tag{3.3.38}$$

As in section 3.3.3 we consider two specific solutions. The first one consists of setting $s_{1,4} = s_{2,2} = s_{5,2} = 0$. This gives the singular fiber $I_5^{ns(012)}$

$$I_5^{ns(012)} \,:\, (5,3,0,3,0,0,0,0) . \tag{3.3.39}$$

Upon imposing the perfect square condition $s_{6,0}^2 - 4s_{3,0}s_{8,0} = p^2$ we find the singular fiber $I_5^{s(012)}$. Alternatively, we set $s_{1,4} = 0$ and we solve non-canonically, as in appendix A.1, the resulting three-term polynomial polynomial

$$P|_{(s_{1,4}=0)} = s_{2,2}^2 s_{8,0} - s_{2,2}s_{6,0}s_{5,2} + s_{5,2}^2 s_{3,0} . \tag{3.3.40}$$

This gives the non-canonical singular fiber $I_{5,nc}^{s(012)}$

$$I_{5,nc}^{s(012)} \,:\, (5,2,0,2,0,0,0,0) \,, \, [-,\sigma_1\sigma_2,\sigma_2\sigma_5,\sigma_1\sigma_3,\sigma_2\sigma_4+\sigma_3\sigma_5,-,\sigma_3\sigma_4,-] . \tag{3.3.41}$$

The second obstruction that we encounter is, again, the seven-term polynomial in the discriminant of the singular $I_4^{ns(01||2)}$

$$\begin{aligned}\Delta \supset P = s_{3,2}^2 s_{5,0}^2 + s_{7,0}(s_{2,2}^2 s_{5,0} - s_{1,2}s_{2,2}s_{6,0} + s_{1,2}^2 s_{7,0}) + \\ + s_{3,2}(-s_{2,2}s_{5,0}s_{6,0} + s_{1,2}(s_{6,0}^2 - 2s_{5,0}s_{7,0})) .\end{aligned} \tag{3.3.42}$$

The canonical solution that we consider requires $s_{1,2} = s_{2,2} = s_{3,2} = 0$. This gives a singular $I_6^{ns(01|||2)}$

$$I_6^{ns(01|||2)} \,:\, (3,3,3,0,0,0,0,0) . \tag{3.3.43}$$

The split version $I_6^{s(01|||2)}$ is found upon imposing that $s_{6,0}^2 - 4s_{5,0}s_{7,0}$ is a perfect, non-zero square. We can also consider the solution where $s_{1,2} = 0$ and the three-term polynomial component of the resulting polynomial is solved non-canonically. This enhancement now produces an $I_{5,nc}^{s(01||2)}$, but this is just a non-generic specialization of one of the $I_{5,nc^2}^{s(01||2)}$ fibers also found in the algorithm and so we do not consider it further.

3.3.11 Split/semi-split Distinction

The split/semi-split distinction arises for singular fibers of Kodaira type I_0^*. The example we provide concerns the possible enhancement of the canonical type $IV^{s(01|2)}$, which was found schematically by

$$I_2^{(01|2)} \longrightarrow I_3^{s(01|2)} \longrightarrow IV^{s(01|2)}\,. \qquad (3.3.44)$$

The discriminant takes a rather simple form

$$\Delta = s_{2,1}s_{7,0}s_{8,0}z^4 + \mathcal{O}(z^5)\,. \qquad (3.3.45)$$

The enhancement we will consider here is when $s_{2,1}=0$. As a consequence $z^2 \mid \mathfrak{s}_2$ and $z^6 \mid \Delta$. This way we have found the semi-split $I_0^{*ss(01|2)}$

$$I_0^{*ss(01|2)} \,:\, (2,2,1,1,1,0,0,0)\,. \qquad (3.3.46)$$

In order for one of the curves of the $I_0^{*ss(01|2)}$ to split into two separate non-intersecting components, we need to satisfy a perfect square condition for the quantity $s_{5,1}^2 - 4s_{1,2}s_{8,0}$. Following appendix A.1 we find the split $I_{0,nc}^{*s(01|2)}$

$$I_{0,nc}^{*s(01|2)} \,:\, (2,2,1,1,1,0,0,0)\,,\; [\sigma_1\sigma_3, -, -, \sigma_1\sigma_2+\sigma_3\sigma_4, -, -, \sigma_2\sigma_4, -]\,. \qquad (3.3.47)$$

3.3.12 A thrice non-canonical I_5

In this section we provide details for an $I_{5,nc^3}^{s(0|1||2)}$. This singular fiber is observed in the algorithm by schematically enhancing

$$I_2^{(01|2)} \longrightarrow I_{3,nc}^{s(0|1|2)} \longrightarrow I_{4,nc^2}^{s(0|1|2)} \longrightarrow I_{5,nc^3}^{s(0|1||2)}\,. \qquad (3.3.48)$$

All the three arrows represent non-canonical enhancements. In particular enhancing from $I_2^{(01|2)}$ to $I_{3,nc}^{s(0|1|2)}$ requires solving a three-term polynomial present in the discriminant. This is $\Delta \supset (s_{8,0}^2 s_{7,0} - s_{8,0}s_{6,0}s_{9,0} + s_{9,0}^2 s_{5,0})$, which is solved by requiring

$$s_{8,0}=\sigma_1\sigma_2,\quad s_{9,0}=\sigma_1\sigma_3,\quad s_{7,0}=\sigma_3\sigma_4,\quad s_{5,0}=\sigma_2\sigma_5,\quad s_{6,0}=\sigma_2\sigma_4+\sigma_3\sigma_5\,. \qquad (3.3.49)$$

Note that this solution implies that (σ_2,σ_3) are coprime. This gives an $I_{3,nc}^{s(0|1|2)}$. Looking at the discriminant of this singular fiber we see that one of the components is $\Delta \supset (\sigma_3^2 s_{1,1} - \sigma_2\sigma_3 s_{2,1} + \sigma_2^2 s_{3,1})$. We apply again the same solution to this three-term polynomial

$$\sigma_3=\xi_2,\quad \sigma_2=\xi_3,\quad s_{1,1}=\xi_3\xi_4,\quad s_{3,1}=\xi_2\xi_5,\quad s_{2,1}=\xi_2\xi_4+\xi_3\xi_5\,. \qquad (3.3.50)$$

Where we used that (σ_2,σ_3) are coprime to set $\xi_1=1$. We have now enhanced the singular fiber to an $I_{4,nc^2}^{s(0|1|2)}$. To obtain the thrice non-canonical I_5 we now consider the two-term polynomial contained in the discriminant at fourth order: $\Delta \supset (\sigma_4\xi_4 - \sigma_5\xi_5)$. Applying the non-canonical solution in appendix A.1

$$\sigma_4 = \delta_1\delta_2, \quad \xi_4 = \delta_3\delta_4, \quad \sigma_5 = \delta_1\delta_3, \quad \xi_5 = \delta_2\delta_4\,. \tag{3.3.51}$$

We have now reached the singular fiber $I_{5,nc^3}^{s(0|1||2)}$

$$\begin{gathered} I_{5,nc^3}^{s(0|2||1)} \,:(1,1,1,0,0,0,0,0)\,, \\ [\xi_3\delta_3\delta_4, \delta_4(\delta_3\xi_2+\delta_2\xi_3), \xi_2\delta_2\delta_4, \xi_3\delta_1\delta_3, \delta_1(\delta_2\xi_3+\delta_3\xi_2), \delta_1\delta_2\xi_2, \sigma_1\xi_3, \sigma_1\xi_2]\,. \end{gathered} \tag{3.3.52}$$

3.4 $U(1)$ Charges of $SU(5)$ Fibers

In section 3.3 a variety of different, canonical and non-canonical, I_5 type singular fibers were found, and are listed in table 3.3. As elliptic fibrations with $SU(5)$ singular fibers are phenomenologically interesting in this section the $U(1)$ charges of the matter loci are determined for the I_5 fibers obtained, which lie in the chosen lop-equivalence class. The $U(1)$ charges are calculated from the intersection number of the matter curve with the Shioda mapped rational section, as explained in section 3.2.3. For the canonical I_5 singular fibers we find, as expected, the same results that were found from the study of toric tops. Details of the relationship between the canonical models and the $SU(5)$ top models and their charges as found in ([1]) are given. In the algorithm a number of non-canonical models which, as far as the authors are aware have not been seen before, were found, some of which are can realize two or even three distinctly charged **10** matter curves, potentially a desirable feature, also some models realize as many as seven differently charged **5** matter curves, which are of some interest in light of ([53]).

3.4.1 Canonical I_5 Models

The $U(1)$ charges of the canonical models are found in table 3.5. Models with these particular $U(1)$ charges are well-studied in the literature. In this subsection we provide a short comparison to the known toric constructions from tops ([59]) , which were constructed with two extra sections in ([1, 24, 51, 54]).

The toric tops as extracted from ([1]) are also given by vanishing orders of the coefficients of the cubic polynomial (3.1.1) and are related to what we called canonical models. In order to see this we need to perform a series of lop transformations to bring them to the equivalence class of singular fibers considered in this chapter. Section 3.2.2 contains the details of the lop transformations. All the tops were found as part of the algorithm and exhaust the canonical models. The $U(1)$ charges of the matter content matched the results found here identically for what was called tops 1 and 2, whereas for tops 3 and 4 a different linear combination of the $U(1)$ charges was used. The details of this linear combination are given in terms of our choice of $U(1)$ generators.

Fiber	Model	Matter Locus	Matter
$I_5^{s(0\|1\|\|2)}$	$(2,2,2,1,0,0,1,0)$	$s_{1,2}$	$\mathbf{5}_{3,1} \oplus \overline{\mathbf{5}}_{-3,-1}$
		$s_{6,0}$	$\mathbf{10}_{1,2} \oplus \overline{\mathbf{10}}_{-1,-2}$
		$s_{7,0}$	$\mathbf{5}_{3,-4} \oplus \overline{\mathbf{5}}_{-3,4}$
		$s_{9,0}$	$\mathbf{5}_{3,6} \oplus \overline{\mathbf{5}}_{-3,-6}$
		$s_{6,0}s_{8,1} - s_{5,1}s_{9,0}$	$\mathbf{5}_{-2,1} \oplus \overline{\mathbf{5}}_{2,-1}$
		$s_{3,2}s_{6,0}^2 - s_{2,2}s_{6,0}s_{7,0} + s_{1,2}s_{7,0}^2$	$\mathbf{5}_{-2,-4} \oplus \overline{\mathbf{5}}_{2,4}$
$I_5^{s(01\|\|2)}$	$(3,2,2,1,0,0,0,0)$	$s_{6,0}$	$\mathbf{10}_{-1,0} \oplus \overline{\mathbf{10}}_{1,0}$
		$s_{7,0}$	$\mathbf{5}_{2,-1} \oplus \overline{\mathbf{5}}_{-2,1}$
		$s_{8,0}$	$\mathbf{5}_{-3,-1} \oplus \overline{\mathbf{5}}_{3,1}$
		$s_{1,3}s_{6,0} - s_{2,2}s_{5,1}$	$\mathbf{5}_{2,0} \oplus \overline{\mathbf{5}}_{-2,0}$
		$s_{3,2}s_{6,0} - s_{2,2}s_{7,0}$	$\mathbf{5}_{-3,0} \oplus \overline{\mathbf{5}}_{3,0}$
		$s_{6,0}s_{9,0} - s_{7,0}s_{8,0}$	$\mathbf{5}_{2,1} \oplus \overline{\mathbf{5}}_{-2,-1}$
$I_5^{s(0\|1\|2)}$	$(3,2,1,1,0,0,1,0)$	$s_{3,1}$	$\mathbf{5}_{-3,1} \oplus \overline{\mathbf{5}}_{3,-1}$
		$s_{6,0}$	$\mathbf{10}_{-1,2} \oplus \overline{\mathbf{10}}_{1,-2}$
		$s_{7,0}$	$\mathbf{5}_{2,-4} \oplus \overline{\mathbf{5}}_{-2,4}$
		$s_{9,0}$	$\mathbf{5}_{2,6} \oplus \overline{\mathbf{5}}_{-2,-6}$
		$s_{5,1}s_{9,0} - s_{6,0}s_{8,1}$	$\mathbf{5}_{2,1} \oplus \overline{\mathbf{5}}_{-2,-1}$
		$s_{3,1}s_{5,1}^2 - s_{2,2}s_{5,1}s_{6,0} + s_{1,3}s_{6,0}^2$	$\mathbf{5}_{-3,-4} \oplus \overline{\mathbf{5}}_{3,4}$
$I_5^{s(01\|2)}$	$(4,2,1,2,0,0,0,0)$	$s_{3,1}$	$\mathbf{5}_{4,0} \oplus \overline{\mathbf{5}}_{-4,0}$
		$s_{6,0}$	$\mathbf{10}_{-2,0} \oplus \overline{\mathbf{10}}_{2,0}$
		$s_{7,0}$	$\mathbf{5}_{-1,1} \oplus \overline{\mathbf{5}}_{1,-1}$
		$s_{8,0}$	$\mathbf{5}_{4,1} \oplus \overline{\mathbf{5}}_{-4,-1}$
		$s_{6,0}s_{9,0} - s_{7,0}s_{8,0}$	$\mathbf{5}_{-1,-1} \oplus \overline{\mathbf{5}}_{1,1}$
		$s_{1,4}s_{6,0}^2 - s_{2,2}s_{5,2}s_{6,0} + s_{2,2}^2s_{8,0}$	$\mathbf{5}_{-1,0} \oplus \overline{\mathbf{5}}_{1,0}$

Table 3.5: $U(1)$ charges of the canonical I_5 models from table 3.3.

In table 3.6 the tops are listed with the numbering and vanishing orders as in appendix A of ([1])(polygon 5), the lop equivalent models as found in Tate's algorithm and details for the linear combination of the $U(1)$ charges for top 3 and top 4.

Top	Fiber	Vanishing Orders	Lop-equivalent model	$U(1)$ Linear Combination
Top 1	$I_5^{s(0\|1\|2)}$	$(2,2,2,0,0,1,0,0)$	$(3,2,1,1,0,0,1,0)$	-
Top 2	$I_5^{s(0\|1\|\|2)}$	$(1,2,3,0,0,1,0,0)$	$(2,2,2,1,0,0,1,0)$	-
Top 3	$I_5^{s(01\|\|2)}$	$(1,1,2,0,0,2,0,0)$	$(3,2,2,1,0,0,0,0)$	$u_1 = -\mathrm{w}_1$ $u_2 = \frac{1}{5}(\mathrm{w}_2 - \mathrm{w}_1)$
Top 4	$I_5^{s(01\|2)}$	$(1,1,1,0,0,2,0,1)$	$(4,2,1,2,0,0,0,0)$	$u_1 = -\mathrm{w}_1$ $u_2 = \frac{1}{5}(\mathrm{w}_2 - \mathrm{w}_1)$

Table 3.6: The lop-equivalent models of the four tops from ([1]). The linear combination of the $U(1)$ charges gives the charges found in table 3.5, u_1 and u_2, in terms of the $U(1)$ charges of the top model, w_1 and w_2. The reason the charges of tops 3 and 4 differ is because the lop translation involves the $\mathbb{Z}_2$ symmetry, which exchanges two of the marked points.

3.4.2 Non-canonical I_5 Models

Listed in tables 3.9 to 3.11 are the $U(1)$ charges of the, respectively once, twice, and thrice, non-canonical I_5 models found in the algorithm. The $U(1)$ charge generators are given by the Shioda map, as described in section 3.2.3, where the zero-section of the fibration corresponds to the divisor $l_1 = 0$ after the $\mathbb{P}^2$ fibration ambient space has been blown up into dP_2. As opposed to the canonical models the majority of the models tabulated in this section were previously unknown. Some of these models appear to have interesting properties for phenomenology, such as the above noted multiple differently charged **10** and **5** curves.

While Tate's algorithm provides a generic procedure there are some caveats that were introduced in the application of it studied in this chapter. There are situations where we were not able to solve for the enhancement locus in the discriminant to a reasonable degree of generality. In these cases we have sometimes, as discussed in section 3.3, used a less generic solution where it was obtainable in such a way that it did not lead to obvious irregularities with the model. In cases where no such solution was obtained we have left that particular subbranch of the Tate tree unexplored.

Throughout the application of Tate's algorithm the fibrations have remained inside the chosen lop-equivalence class and so each each model in these tables then represents an entire lop-orbit of fibrations. The $\mathbb{Z}_2$ symmetry which acts inside this orbit interchanges two of the three marked points of the fibration, which correspond, in the dP_2 hypersurface, to the exchange of l_1 and l_2. As the $U(1)$ charges are computed from a Shioda map where the zero-section is taken to be $l_1 = 0$ the $U(1)$ charges are rewritten as a linear combination under this symmetry in an identical manner to the linear combinations occurring in the tops in table 3.6.

One may point out the surprising paucity of non-minimal matter loci in these models with highly specialised coefficients. In the fibrations which are at least twice non-canonical there can occur polynomial enhancement loci where some of the terms in the solutions (as given in appendix A.1) are fixed by a coprimality condition coming from a previously solved polynomial. Were these terms not fixed to unity by the algorithm then they would contribute non-minimal loci to the fibrations.

In ([1, 51]) there were listed tops corresponding to an $SU(4)$ non-abelian singularity with two additional rational sections, and it was noted that one expects multiple **10** matter curves where these tops are specialized with some non-generic coefficients of the defining polynomial, and such a model, which realizes multiple **10** curves, was constructed there from the $SU(4)$ tops. Included in table 3.7 are the relations (via the lops) between these $SU(4)$ tops and the $SU(4)$ canonical models which underlie the once non-canonical $SU(5)$ models obtained in the algorithm.

Top Model	Fiber	Vanishing Orders	Lop-equivalent Model
Top 1	$I_4^{(0\|1\|2)}$	$(1,1,2,0,0,1,0,0)$	$(2,1,1,1,0,0,1,0)$
Top 2	$I_4^{(01\|\|2)}$	$(0,1,2,0,0,2,0,0)$	$(2,2,2,1,0,0,0,0)$
Top 3	$I_4^{(01\|2)}$	$(1,1,1,0,0,1,0,1)$	$(3,2,1,1,0,0,0,0)$
Top 4	$I_4^{(01\|\|2)}$	$(0,1,2,0,0,1,0,1)$	$(2,2,2,1,0,0,0,0)$
Top 5	$I_4^{(01\|2)}$	$(0,0,1,0,0,2,0,1)$	$(3,2,1,1,0,0,0,0)$

Table 3.7: The $SU(4)$ tops associated to polygon 5 in appendix B of ([1]) are related to the canonical I_4 models listed in table 3.2 by lop-equivalence.

Note that for $SU(4)$ top 4 is lop equivalent to top 2 and top 5 is lop equivalent top 3, and their $U(1)$ charges, as listed in appendix B of ([1]), can be written as a linear combination of the lop-equivalent model.

Fiber	Model	Matter Locus	Matter
$I_5^{s(0\|1\|\|2)}$	$(2,2,2,0,0,0,0,0)$ $[-,-,-,\sigma_2\sigma_5,\sigma_2\sigma_4+\sigma_3\sigma_5,\sigma_3\sigma_4,\sigma_1\sigma_2,\sigma_1\sigma_3]$	σ_1	$\mathbf{5}_{-3,-6}\oplus\overline{\mathbf{5}}_{3,6}$
		σ_3	$\mathbf{5}_{2,-6}\oplus\overline{\mathbf{5}}_{-2,6}$
		σ_4	$\mathbf{5}_{-3,4}\oplus\overline{\mathbf{5}}_{3,-4}$
		$\sigma_2\sigma_4-\sigma_3\sigma_5$	$\mathbf{10}_{-1,-2}\oplus\overline{\mathbf{10}}_{1,2}$
		$(A.2.1)$	$\mathbf{5}_{-3,-1}\oplus\overline{\mathbf{5}}_{3,1}$
		$(A.2.2)$	$\mathbf{5}_{2,-1}\oplus\overline{\mathbf{5}}_{-2,1}$
		$(A.2.3)$	$\mathbf{5}_{2,4}\oplus\overline{\mathbf{5}}_{-2,-4}$
$I_5^{s(0\|1\|2)}$	$(2,1,1,1,0,0,1,0)$ $[\sigma_1\sigma_3,\sigma_1\sigma_2,-,\sigma_3\sigma_4,\sigma_2\sigma_4,-,-,-]$	σ_2	$\mathbf{10}_{1,-2}\oplus\overline{\mathbf{10}}_{-1,2}$
		σ_4	$\mathbf{10}_{1,3}\oplus\overline{\mathbf{10}}_{-1,-3}$
		$s_{7,0}$	$\mathbf{5}_{-2,4}\oplus\overline{\mathbf{5}}_{2,-4}$
		$s_{9,0}$	$\mathbf{5}_{-2,-6}\oplus\overline{\mathbf{5}}_{2,6}$
		$\sigma_4 s_{3,1}-\sigma_1 s_{7,0}$	$\mathbf{5}_{3,-1}\oplus\overline{\mathbf{5}}_{-3,1}$
		$\sigma_2 s_{8,1}-\sigma_3 s_{9,0}$	$\mathbf{5}_{3,4}\oplus\overline{\mathbf{5}}_{-3,-4}$
		$(A.2.4)$	$\mathbf{5}_{-2,-1}\oplus\overline{\mathbf{5}}_{2,1}$
$I_5^{s(0\|1\|\|2)}$	$(2,1,1,1,0,0,1,0)$ $[-,\sigma_1\sigma_2,\sigma_1\sigma_3,-,\sigma_2\sigma_4,\sigma_3\sigma_4,-,-]$	σ_2	$\mathbf{10}_{-1,-2}\oplus\overline{\mathbf{10}}_{1,2}$
		σ_3	$\mathbf{5}_{-3,4}\oplus\overline{\mathbf{5}}_{3,-4}$
		σ_4	$\mathbf{10}_{-1,3}\oplus\overline{\mathbf{10}}_{1,-3}$
		$s_{9,0}$	$\mathbf{5}_{-3,-6}\oplus\overline{\mathbf{5}}_{3,6}$
		$\sigma_4 s_{1,2}-\sigma_1 s_{5,1}$	$\mathbf{5}_{-3,-1}\oplus\overline{\mathbf{5}}_{3,1}$
		$\sigma_2\sigma_4 s_{8,1}-s_{5,1}s_{9,0}$	$\mathbf{5}_{2,4}\oplus\overline{\mathbf{5}}_{-2,-4}$
		$(A.2.5)$	$\mathbf{5}_{2,-1}\oplus\overline{\mathbf{5}}_{-2,1}$
$I_5^{s(1\|0\|2)}$	$(3,2,1,1,0,0,0,0)$ $[-,-,-,-,\sigma_1\sigma_2,\sigma_1\sigma_3,\sigma_2\sigma_4,\sigma_3\sigma_4]$	σ_1	$\mathbf{10}_{-2,2}\oplus\overline{\mathbf{10}}_{2,-2}$
		σ_2	$\mathbf{10}_{3,2}\oplus\overline{\mathbf{10}}_{-3,-2}$
		σ_3	$\mathbf{5}_{-1,6}\oplus\overline{\mathbf{5}}_{1,-6}$
		σ_4	$\mathbf{5}_{4,6}\oplus\overline{\mathbf{5}}_{-4,-6}$
		$s_{3,1}$	$\mathbf{5}_{4,1}\oplus\overline{\mathbf{5}}_{-4,-1}$
		$(A.2.6)$	$\mathbf{5}_{-1,1}\oplus\overline{\mathbf{5}}_{1,-1}$
		$(A.2.7)$	$\mathbf{5}_{-1,-4}\oplus\overline{\mathbf{5}}_{1,4}$

Table 3.8: $U(1)$ charges of the once non-canonical I_5 models from table 3.3.

Fiber	Model	Matter Locus	Matter
$I_5^{s(01\|2)}$	$(3,2,1,1,0,0,0,0)$ $[\sigma_2\sigma_5, \sigma_2\sigma_4+\sigma_3\sigma_5, \sigma_3\sigma_4, \sigma_1\sigma_2, \sigma_1\sigma_3, -, -, -]$	σ_1	$\mathbf{10}_{2,0} \oplus \overline{\mathbf{10}}_{-2,0}$
		σ_3	$\mathbf{10}_{-3,0} \oplus \overline{\mathbf{10}}_{3,0}$
		σ_4	$\mathbf{5}_{-4,0} \oplus \overline{\mathbf{5}}_{4,0}$
		$s_{7,0}$	$\mathbf{5}_{1,-1} \oplus \overline{\mathbf{5}}_{-1,1}$
		$s_{8,0}$	$\mathbf{5}_{-4,-1} \oplus \overline{\mathbf{5}}_{4,1}$
		$\sigma_1\sigma_3 s_{9,0} - s_{7,0}s_{8,0}$	$\mathbf{5}_{1,1} \oplus \overline{\mathbf{5}}_{-1,-1}$
		$(A.2.8)$	$\mathbf{5}_{1,0} \oplus \overline{\mathbf{5}}_{-1,0}$
$I_5^{s(1\|02)}$	$(4,2,0,2,0,0,0,0)$ $[-, -, \sigma_3\sigma_4, -, \sigma_2\sigma_4+\sigma_3\sigma_5, \sigma_1\sigma_3, \sigma_2\sigma_5, \sigma_1\sigma_2]$	σ_1	$\mathbf{5}_{0,6} \oplus \overline{\mathbf{5}}_{0,-6}$
		σ_2	$\mathbf{5}_{1,6} \oplus \overline{\mathbf{5}}_{-1,-6}$
		σ_3	$\mathbf{5}_{-1,1} \oplus \overline{\mathbf{5}}_{1,-1}$
		σ_4	$\mathbf{5}_{1,1} \oplus \overline{\mathbf{5}}_{-1,-1}$
		σ_5	$\mathbf{5}_{-1,-4} \oplus \overline{\mathbf{5}}_{1,4}$
		$\sigma_2\sigma_4 - \sigma_3\sigma_5$	$\mathbf{10}_{0,2} \oplus \overline{\mathbf{10}}_{0,-2}$
		$(A.2.9)$	$\mathbf{5}_{0,-4} \oplus \overline{\mathbf{5}}_{0,4}$
		$(A.2.10)$	$\mathbf{5}_{0,1} \oplus \overline{\mathbf{5}}_{0,-1}$
$I_5^{s(012)}$	$(5,2,0,2,0,0,0,0)$ $[-, \sigma_1\sigma_2, \sigma_2\sigma_5, \sigma_1\sigma_3, \sigma_2\sigma_4+\sigma_3\sigma_5, -, \sigma_3\sigma_4, -]$	σ_2	$\mathbf{5}_{-1,0} \oplus \overline{\mathbf{5}}_{1,0}$
		σ_3	$\mathbf{5}_{1,1} \oplus \overline{\mathbf{5}}_{-1,-1}$
		σ_4	$\mathbf{5}_{-1,-1} \oplus \overline{\mathbf{5}}_{1,1}$
		σ_5	$\mathbf{5}_{1,0} \oplus \overline{\mathbf{5}}_{-1,0}$
		$\sigma_2\sigma_4 - \sigma_3\sigma_5$	$\mathbf{10}_{0,0} \oplus \overline{\mathbf{10}}_{0,0}$
		$\sigma_3 s_{7,0} - \sigma_2 s_{9,0}$	$\mathbf{5}_{0,-1} \oplus \overline{\mathbf{5}}_{0,1}$
		$\sigma_4 s_{7,0} - \sigma_5 s_{9,0}$	$\mathbf{5}_{0,1} \oplus \overline{\mathbf{5}}_{0,-1}$
		$(A.2.11)$	$\mathbf{5}_{0,0} \oplus \overline{\mathbf{5}}_{0,0}$

Table 3.9: $U(1)$ charges of the once non-canonical I_5 models from table 3.3 (continued).

Fiber	Model	Matter Locus	Matter
$I_5^{s(01\|\|2)}$	$(2,2,2,0,0,0,0,0)$ $[\xi_3\xi_4, \xi_2\xi_4+\xi_3\xi_5, \xi_2\xi_5, \sigma_3\xi_3, \sigma_2\xi_3+\sigma_3\xi_2, \sigma_2\xi_2, -, -]$	ξ_2	$\mathbf{5}_{3,-1} \oplus \overline{\mathbf{5}}_{-3,1}$
		σ_2	$\mathbf{5}_{-2,1} \oplus \overline{\mathbf{5}}_{2,-1}$
		$\sigma_2\xi_3 - \sigma_3\xi_2$	$\mathbf{10}_{1,0} \oplus \overline{\mathbf{10}}_{-1,0}$
		$\sigma_2\xi_4 - \sigma_3\xi_5$	$\mathbf{5}_{3,0} \oplus \overline{\mathbf{5}}_{-3,0}$
		$\xi_2 s_{8,0} - \xi_3 s_{9,0}$	$\mathbf{5}_{3,1} \oplus \overline{\mathbf{5}}_{-3,-1}$
		$\sigma_2 s_{8,0} - \sigma_3 s_{9,0}$	$\mathbf{5}_{-2,-1} \oplus \overline{\mathbf{5}}_{2,1}$
		$(A.2.12)$	$\mathbf{5}_{-2,0} \oplus \overline{\mathbf{5}}_{2,0}$
$I_5^{s(0\|1\|\|2)}$	$(2,1,1,1,0,0,0,0)$ $[-, \sigma_1\xi_3, \sigma_1\xi_2, -, \sigma_4\xi_3, \sigma_4\xi_2, \xi_3\xi_4, \xi_2\xi_4]$	ξ_2	$\mathbf{5}_{2,-6} \oplus \overline{\mathbf{5}}_{-2,6}$
		ξ_3	$\mathbf{10}_{-1,-2} \oplus \overline{\mathbf{10}}_{1,2}$
		ξ_4	$\mathbf{5}_{-3,-6} \oplus \overline{\mathbf{5}}_{3,6}$
		σ_4	$\mathbf{10}_{-1,3} \oplus \overline{\mathbf{10}}_{1,-3}$
		$(A.2.13)$	$\mathbf{5}_{2,-1} \oplus \overline{\mathbf{5}}_{-2,1}$
		$(A.2.14)$	$\mathbf{5}_{-3,-1} \oplus \overline{\mathbf{5}}_{3,1}$
		$(A.2.15)$	$\mathbf{5}_{2,4} \oplus \overline{\mathbf{5}}_{-2,-4}$
$I_5^{s(01\|\|2)}$	$(2,1,1,1,0,0,0,0)$ $[\xi_3\xi_4, \sigma_2\xi_3, \sigma_3\xi_3, \xi_2\xi_4+\xi_3\xi_5, \sigma_2\xi_2, \sigma_3\xi_2, \xi_2\xi_5, -]$	ξ_2	$\mathbf{10}_{1,1} \oplus \overline{\mathbf{10}}_{-1,-1}$
		ξ_5	$\mathbf{5}_{3,1} \oplus \overline{\mathbf{5}}_{-3,-1}$
		σ_2	$\mathbf{5}_{-2,1} \oplus \overline{\mathbf{5}}_{2,-1}$
		σ_3	$\mathbf{10}_{1,0} \oplus \overline{\mathbf{10}}_{-1,0}$
		$\xi_2\xi_5\sigma_2 - \sigma_3 s_{9,0}$	$\mathbf{5}_{-2,-1} \oplus \overline{\mathbf{5}}_{2,1}$
		$(A.2.16)$	$\mathbf{5}_{3,0} \oplus \overline{\mathbf{5}}_{-3,0}$
		$(A.2.17)$	$\mathbf{5}_{-2,0} \oplus \overline{\mathbf{5}}_{2,0}$
$I_5^{s(1\|0\|2)}$	$(2,1,1,1,0,0,0,0)$ $[\sigma_3\sigma_4, \sigma_3\xi_1\xi_3, -, \sigma_2\sigma_4+\sigma_3\xi_1\xi_2, \sigma_2\xi_1\xi_3, \xi_3\xi_4, \sigma_2\xi_1\xi_2, \xi_2\xi_4]$	ξ_1	$\mathbf{10}_{3,2} \oplus \overline{\mathbf{10}}_{-3,-2}$
		ξ_2	$\mathbf{5}_{4,6} \oplus \overline{\mathbf{5}}_{-4,-6}$
		ξ_3	$\mathbf{10}_{-2,2} \oplus \overline{\mathbf{10}}_{2,-2}$
		ξ_4	$\mathbf{5}_{-1,6} \oplus \overline{\mathbf{5}}_{1,-6}$
		σ_2	$\mathbf{10}_{-2,-3} \oplus \overline{\mathbf{10}}_{2,3}$
		$\xi_3\xi_4\sigma_3 - \sigma_2 s_{3,1}$	$\mathbf{5}_{4,1} \oplus \overline{\mathbf{5}}_{-4,-1}$
		$(A.2.18)$	$\mathbf{5}_{-1,-4} \oplus \overline{\mathbf{5}}_{1,4}$
		$(A.2.19)$	$\mathbf{5}_{-1,1} \oplus \overline{\mathbf{5}}_{1,-1}$
$I_5^{s(0\|2\|\|1)}$	$(1,1,1,1,0,0,1,0)$ $[\xi_2\xi_3\xi_5\xi_6, \xi_3\xi_6\sigma_4+\xi_2\xi_5\sigma_3, \sigma_3\sigma_4,$ $\xi_2\xi_6\xi_7+\xi_3\xi_4\xi_5, \xi_1\xi_2\xi_3\xi_6, \xi_1\xi_2\sigma_3, \xi_4\xi_7, \xi_1\xi_3\xi_4]$	ξ_3	$\mathbf{10}_{3,-1} \oplus \overline{\mathbf{10}}_{-3,1}$
		ξ_4	$\mathbf{5}_{4,2} \oplus \overline{\mathbf{5}}_{-4,-2}$
		ξ_5	$\mathbf{10}_{3,4} \oplus \overline{\mathbf{10}}_{-3,-4}$
		ξ_6	$\mathbf{10}_{3,4} \oplus \overline{\mathbf{10}}_{-3,-4}$
		ξ_8	$\mathbf{5}_{4,7} \oplus \overline{\mathbf{5}}_{-4,-7}$
		σ_3	$\mathbf{5}_{4,-3} \oplus \overline{\mathbf{5}}_{-4,3}$
		$(A.2.20)$	$\mathbf{5}_{-1,-3} \oplus \overline{\mathbf{5}}_{1,3}$
		$(A.2.21)$	$\mathbf{5}_{-1,2} \oplus \overline{\mathbf{5}}_{1,-2}$

Table 3.10: $U(1)$ charges of the twice non-canonical I_5 models from table 3.3.

Fiber	Model	Matter Locus	Matter
$I_5^{s(0\|1\|\|2)}$	$(1,1,1,0,0,0,0,0)$ $[\xi_3\delta_3\delta_4, \delta_4(\delta_3\xi_2+\delta_2\xi_3), \xi_2\delta_2\delta_4, \xi_3\delta_1\delta_4, \delta_1(\delta_2\xi_3+\delta_3\xi_2), \delta_1\delta_2\xi_2, \sigma_1\xi_3, \sigma_1\xi_2]$	δ_1	$\mathbf{10}_{1,-3} \oplus \overline{\mathbf{10}}_{-1,3}$
		δ_2	$\mathbf{5}_{3,-4} \oplus \overline{\mathbf{5}}_{-3,4}$
		ξ_2	$\mathbf{5}_{-2,6} \oplus \overline{\mathbf{5}}_{2,-6}$
		σ_1	$\mathbf{5}_{3,6} \oplus \overline{\mathbf{5}}_{-3,-6}$
		$\xi_2\delta_3 - \xi_3\delta_2$	$\mathbf{10}_{1,2} \oplus \overline{\mathbf{10}}_{-1,-2}$
		$(A.2.22)$	$\mathbf{5}_{-2,1} \oplus \overline{\mathbf{5}}_{2,-1}$
		$(A.2.23)$	$\mathbf{5}_{3,1} \oplus \overline{\mathbf{5}}_{-3,-1}$
		$(A.2.24)$	$\mathbf{5}_{-2,-4} \oplus \overline{\mathbf{5}}_{2,4}$

Table 3.11: $U(1)$ charges of the single thrice non-canonical I_5 model from table 3.3.

3.5 Exceptional Singular Fibers

In this section the algorithm is continued up to the exceptional singular fibers. In determining the exceptional fibers we recall that the sections can only intersect the fiber components of multiplicity one, which means that there is a very restricted number of singular fibers.

For what concerns the type IV^* singular fiber there are three different ways in which the sections can intersect the multiplicity one components. These are the types $IV^{*(012)}, IV^{*(01|2)}$ and $IV^{*(0|1|2)}$. As can be seen from figure 3.6 the three multiplicity one components of the IV^* singular fiber appear symmetrically, and so sections separated by a slash merely indicates that they do not intersect the same multiplicity one component.

Regarding the singular III^* fibers, the possible ways the sections can intersect the components restrict the range of singular fibers to $III^{*(012)}$ and $III^{*(01|2)}$. The different singular fibers can be seen in figure 3.7.

Finally it is clear that the only type II^* fiber one could find (since there is only one multiplicity one component) is the $II^{*(012)}$. This fiber is also shown in figure 3.7.

It was also possible to obtain the singular fibers corresponding to gauge groups G_2 and F_4 which come from, respectively, the non-split singular fiber types $I_0^{*ns(012)}$ and $IV^{*ns(012)}$.

Proceeding through these subbranchs of the Tate tree will involve the I_n^* fibers corresponding to Dynkin diagrams of D-type in the split case. There fibers are composed of a chain of multiplicity two nodes with two multiplicity one nodes connected to each end of the chain. As the rational sections can only intersect the multiplicity one nodes they are

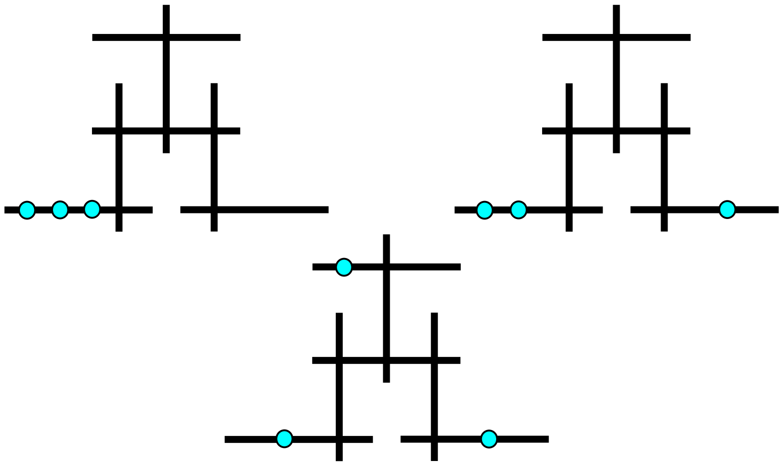


Figure 3.6: The type IV^{*s} fibers. The sections, which intersect the components of the IV^{*s} fiber represented by the blue nodes, are seen to intersect only the external, multiplicity one components. Because of the S_3 symmetry we write these as $IV^{*s(ijk)}$, $IV^{*s(ij|k)}$, and $IV^{*s(i|j|k)}$ respectively.

constrained to lie of these outer legs. The notation of these fibers shall be (01) represents two sections on the same leg, (0|1) represents two section intersecting two of the outer legs attached to the same end of the chain, and (0||1) will represent two sections sitting on multiplicity one component separated by the length of the chain.

3.5.1 Canonical Enhancements to Exceptional Singular Fibers

The starting point for the enhancements to the possible canonical exceptional singular fibers is the $I_0^{*ss(01|2)}$: $(2, 2, 1, 1, 1, 0, 0, 0)$. Recall that one of the fiber components will split only if the condition $s_{5,1}^2 - 4s_{1,2}s_{8,0} = p^2$ is satisfied for some p. The discriminant at sixth order takes the form

$$\Delta = s_{7,0}^2 s_{8,0}^2 (s_{5,1}^2 - 4s_{1,2}s_{8,0})(s_{1,2}s_{7,0}^2 - s_{3,1}s_{5,1}s_{7,0} + s_{3,1}^2 s_{8,0})^2)z^6 + \mathcal{O}(z^7)\,. \tag{3.5.1}$$

First let $z \mid \mathfrak{s}_8$ and the resulting fiber is of type $I_1^{s(0|2||1)}$. The discriminant at seventh order reads

$$\Delta = s_{5,1}^3 s_{7,0}^3 (s_{3,1}s_{5,1} - s_{1,2}s_{7,0})^2 s_{9,0}^2 z^7 + \mathcal{O}(z^8)\,. \tag{3.5.2}$$

Now let $z^2 \mid \mathfrak{s}_5$ and the first exceptional singular fiber is found; it is of type $IV^{*(0|1|2)}$

$$IV^{*(0|1|2)} : \quad (2, 2, 1, 2, 1, 0, 1, 0)\,. \tag{3.5.3}$$

This subbranch of the tree does not continue because the discriminant now takes the form $\Delta = s_{1,2}s_{7,0}s_{9,0}z^8 + \mathcal{O}(z^9)$ and the only possible enhancement that remains inside

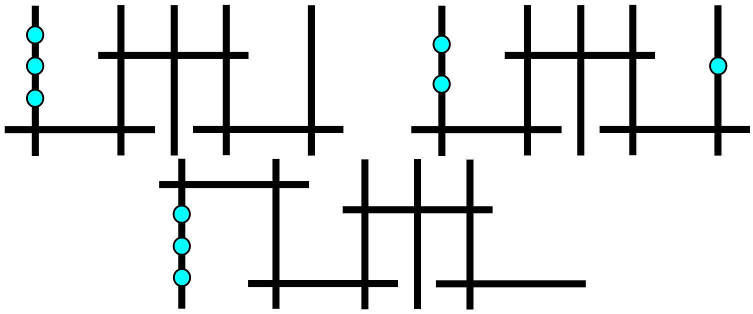


Figure 3.7: The type III^* and II^* fibers. Shown are the two type $III^{*(ijk)}$ and $III^{*(ij|k)}$ fibers where the sections are distributed over the two multiplicity one components, and the single type $II^{*(ijk)}$ fiber, which has all three sections intersecting the single multiplicity one component.

the lop-equivalence class, the vanishing of $s_{1,2}$, is a non-minimal enhancement.

Looking back at the I_0^* starting point, the discriminant can instead be enhanced by letting the three-term polynomial vanish, through the canonical solution $s_{1,2} = s_{3,1} = 0$. This gives an $I_1^{*s(01||2)}$ singular fiber. The discriminant at seventh order takes the form

$$\Delta = s_{2,2}^2 s_{5,1}^3 s_{7,0}^5 s_{8,0}^2 z^7 + \mathcal{O}(z^8)\,. \tag{3.5.4}$$

The discriminant is enhanced further by letting $s_{5,1} = 0$. This gives the second exceptional singular fiber, that is a type $IV^{*(01|2)}$

$$IV^{*(01|2)}\ :\quad (3,2,2,2,1,0,0,0)\,. \tag{3.5.5}$$

Proceeding in this subbranch, the discriminant now reads

$$\Delta = s_{2,2}^4 s_{7,0}^4 s_{8,0}^4 z^8 + \mathcal{O}(z^9)\,. \tag{3.5.6}$$

The only enhancement which is possible (as all the others are non-minimal enhancements) is $s_{2,2} = 0$. The canonical excpetional singular fiber that arises from this enhancement is $III^{*(01|2)}$

$$III^{*(01|2)}\ :\quad (3,3,2,2,1,0,0,0)\,. \tag{3.5.7}$$

Every further enhancement in this subbranch is a non-minimal fibration.

3.5.2 Non-canonical Enhancements to Exceptional Singular Fibers

In this section the remaining exceptional fibers are obtained through non-canonical enhancements of the discriminant. The starting point is the singular $I_3^{ns(012)}$ given by the vanishing orders $(3,2,0,2,0,0,0,0)$. The discriminant contains $\Delta \supset (s_{6,0}^2 - 4s_{3,0}s_{8,0})$. Following appendix A.1 it can be solved non-canonically to find a non-split $I_{0,nc}^{*ns(012)}$ associated to gauge group G_2

$$I_{0,nc}^{*ns(012)}: \quad (3,2,0,2,0,0,0,0) \quad [-,-,\mu\sigma_3^2,-,2\mu\sigma_3\sigma_8,-,\mu\sigma_8^2,-]\,. \tag{3.5.8}$$

The next exceptional singular fiber is found through the following series of enhancements

$$\begin{gathered} I_{0,nc}^{*ns(012)} \xrightarrow{\{s_{1,3},s_{2,2},s_{5,2}=0\}} I_{1,nc}^{*ns(012)} \xrightarrow{P=0} IV_{nc^2}^{*ns(012)} \\ P = (\sigma_8^2 s_{3,1} - \sigma_3\sigma_8 s_{6,1} + \sigma_3^2 s_{8,1})\,. \end{gathered} \tag{3.5.9}$$

Where the non-canonical solution to the three-term polynomial was applied to find a singular $IV_{nc^2}^{*ns(012)}$ with gauge group F_4

$$IV_{nc^2}^{*ns(012)}: (4,3,0,3,0,0,0,0) \quad [-,-,\mu\xi_2^2+\xi_2\xi_5 z,-,2\mu\xi_2\xi_3+(\xi_2\xi_4+\xi_3\xi_5)z,-,\mu\xi_3^2+\xi_3\xi_4 z,-]\,. \tag{3.5.10}$$

It was also necessary to specialize terms linear in z in the expansion of the coefficients. From this singular fiber the remaining two fiber types can be reached through

$$\begin{gathered} IV_{nc^2}^{*ns(012)} \xrightarrow{s_{1,4}=0} III_{nc^2}^{*(012)} \xrightarrow{Q=0} II_{nc^3}^{*(012)} \\ Q = (s_{5,3}\xi_2 - s_{2,3}\xi_3)\,. \end{gathered} \tag{3.5.11}$$

The singular fibers obtained this way are type $III_{nc^2}^{*(012)}$

$$III_{nc^2}^{*(012)}: \quad (5,3,0,3,0,0,0,0) \quad [-,-,\mu\xi_2^2+\xi_2\xi_5 z,-,2\mu\xi_2\xi_3+(\xi_2\xi_4+\xi_3\xi_5)z,-,\mu\xi_3^2+\xi_3\xi_4 z,-]\,, \tag{3.5.12}$$

and the singular fiber type $II_{nc^3}^{*(012)}$

$$\begin{gathered} II_{nc^3}^{*(012)}: \quad (5,3,0,3,0,0,0,0) \\ [-,\delta_1\delta_3,\mu\delta_3^2+\delta_3\xi_5 z,\delta_1\delta_2,2\mu\delta_2\delta_3+(\delta_3\xi_4+\delta_2\xi_5)z,-,\mu\delta_2^2+\delta_2\xi_4 z,-]\,. \end{gathered} \tag{3.5.13}$$

Chapter 4

Aspects of M-Theory

In Chapter 2 we encountered M-Theory through a chain of dualities relating it to F-Theory. In this chapter we are going to study it in its own right, as the non-perturbative limit of Type IIA string theory, whose low energy limit is 11-dimensional supergravity. In ([2]) Witten was then able to relate the known string theories to a single 11-dimensional theory, then called M-Theory. The perturbative limit could be re-obtained by compactifying M-Theory on a circle, whose radius was found to be proportional to the string coupling g_s of Type IIA string theory. Through T-duality and S-duality, and thanks to the statement that the low energy limit of M-Theory compactified on an interval $S^1/\mathbb{Z}_2$ is found to be the Heterotic $E_8 \times E_8$ string theory ([86]), it was possible to relate all the known string theories to one higher dimensional theory.

Nevertheless, very little was known about M-Theory and still today a great part of it remains unexplored. The reason behind this, is that perturbation theory has no access to it due to the absence of the string coupling. This fact is indeed signalling that the fundamental string is not part of the theory any more, but is replaced by membranes and fivebranes, also known as M2 and M5-branes. However, the dynamics of such branes is not inferable from that of D-branes exactly for the lack of fundamental strings. While the worldvolume theories on stacks of D-branes can be extracted from open string scatterings, the same does not hold for M-branes. Indeed, M5-branes interact through M2-branes, but the scattering of the latter has not yielded the same results of strings scattering. As a consequence, M-Theory has only a non-perturbative regime, into which it has been hard to gain insight. However, it has provided a unified perspective which has helped to understand many aspects of 10-dimensional string theories and lower dimensional field theories.

In particular, the worldvolume theories on stacks of M2 and M5-branes have been two of the problems of the string theory program which have received most attention in the last years. In this chapter, we review recent progress in the understanding of such theories. The dynamics on parallel M2-branes has advanced considerably in the last years, while the one on parallel M5-branes is still largely unexplored. Both the BLG (Bagger-Lambert-Gustavvson)([3,4]) and the ABJM (Aharony-Bergman-Jafferis-Maldacena) ([5]) models allowed to gain great insight into the theories of coincident M2-branes. On the other hand, a framework in which progress has been achieved towards theories of multiple M5-branes in M-Theory, is the one of higher gauge theory, see e.g. ([87–90]). Such theories are a 'categorification' of usual gauge theory, where the latter are described by a principle bundle over space-time, allowing to define the parallel transport of point-particles. This can be understood in the D-brane picture as the parallel transport of the end-points of the strings through which D-branes interact. However, the mirror picture for membranes in M-Theory requires the parallel transport of string-like objects (since interactions are through M2-branes, whose boundaries are 1-dimensional). It turns out that taking into account the parallel transport of strings requires the 'categorification' to higher gauge structures, i.e. one needs to consider 2-bundles (or higher structures) rather than usual vector bundles. This means considering a functor between two categories rather than a morphism between two objects of a category, i.e. the usual vector bundles in gauge theory. We will not investigate further this direction, but the BLG, ABJM models and the Lambert-Papageorgakis model for M5-branes, which we will discuss later, have been shown to be expressible in the framework of higher gauge theory ([91, 92]).

Nevertheless, the theory on coincident M5 has produced, through different compactifications, a number of results and dualities in lower dimensional field theories that we also review in the following.

4.1 M-Theory and M-Branes

M2 and M5-branes are half BPS solution of 11-dimensional supergravity, the low energy theory describing M-Theory. Recall the field content of 11d supergravity. We have the metric (graviton) $g_{\mu\nu}(x)$, $\mu, \nu = 0, 1, \ldots, 10$ which constitutes 44 degrees of freedom on shell, being a symmetric traceless tensor. The other 84 bosonic degrees of freedom is represented by a three form $C_3 = C_{\mu\nu\lambda}$. The super-partner of the graviton is the gravitino $\Psi_{\mu\alpha}$, transforming in a tensor product representation of a space-time vector and a spinor. On shell, the gravitino contributes indeed the 128 degrees of freedom needed for supersymmetry. The action for 11-dimensional supergravity turns out to be unique and it was found in the seminal paper by Cremmer, Julia and Scherk and its bosonic part is ([93, 94])

$$S_{11} \supset \int d^{11}x\sqrt{|g|}\,R - \left(\frac{1}{2}F\wedge\star F + \frac{1}{6}C_3\wedge F\wedge F\right), \tag{4.1.1}$$

where F is the field strength of the potential C_3 and R is the Ricci scalar of space-time.

The relationship between 11-dimensional supergravity and the 10-dimensional supergravity theories has been elucidated in a number of papers ([95–97]). The 10-dimensional theory turns out to be recovered in the limit in which the 11-dimensional one is studied on a space of the form $M^{10}\times S^1$, and the radius R of the circle vanishes. Such relation between supergravity theories was finally interpreted in the seminal papers by Townsend ([98]) and Witten ([2]). Type IIA string theory, being the UV completion of Type IIA supergravity, is understood as the compactification of a new theory, then called M-Theory, on a circle. In particular, the relation between the radius of the circle and the coupling constant g_s is found to be

$$R = g_s\,l_s, \tag{4.1.2}$$

where l_s is the string length. We see that upon reducing on the circle the perturbative limit in g_s is restored, but in the decompactification limit M-theory results to be the strongly coupled regime of type IIA string theory. In particular we see that in the strongly coupled phase of type IIA an extra dimension opens up. However, 11-dimensional supergravity, and therefore M-theory, do not have a dimensionless parameter about which it would be possible to expand. It follows that, without any perturbative access to it, M-Theory remains greatly unexplored. In the following, we will look at the fundamental objects of M-Theory, which will turn out to be membranes rather than strings.

In order to find the BPS solutions one looks at the gravitino supersymmetry variation ([94])

$$\delta\Psi_{\mu\alpha} = D_\mu(\omega)\epsilon + \frac{i}{288}\left(\Gamma^{\nu\lambda\sigma\tau\omega}{}_{\mu} - 8\Gamma^{\lambda\sigma\tau\omega}\delta^{\nu}_{\mu}\right)F_{\nu\lambda\sigma\tau\omega}\epsilon \equiv 0 \tag{4.1.3}$$

where $D_\mu(\omega)$ is the covariant derivative depending on the spin connection. Solutions were found describing half BPS objects which are now known as M2 and M5-branes. Without going into the details of the actual supergravity solutions, we can get a hint as to why we must have such solutions by looking at the superalgebra associated to 11-dimensional supergravity

$$\{Q_\alpha, Q_\beta\} = (\Gamma^\mu C^{-1})_{\alpha\beta}P_\mu + \frac{1}{2}(\Gamma^{\mu\nu}C^{-1})_{\alpha\beta}Z_{\mu\nu} + \frac{1}{5!}(\Gamma^{\mu\nu\lambda\rho\sigma}C^{-1})Z_{\mu\nu\lambda\rho\sigma}. \tag{4.1.4}$$

The presence of the central charges $Z_{\mu\nu}$ and $Z_{\mu\nu\lambda\rho\sigma}$ signal the existence of respectively two and five dimensional object in space. A similar argument follows from the potential three-form C_3 which couples electrically to the worldvolume $\mathcal{S}$ of M2-branes through $\int_{\mathcal{S}} C_3 \equiv \int_{\mathcal{S}} C_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda$, where we pull-back the C_3 form to the worldvolume of the M2-brane. Similarly we find the magnetic coupling by applying Hodge duality to the field strength of C_3

$$C_3 \longrightarrow F_4 \xrightarrow{\star} F_7 \longrightarrow C_6. \tag{4.1.5}$$

Then we can pull-back C_6 to the worldvolume $\mathcal{W}$ of M5-branes via $\int_{\mathcal{W}} C_6$ to obtain the correct coupling. As we saw, reducing on a circle we recover the perturbative limit of type IIA string theory, which means that by reducing the M2 and the M5-branes we should recover the D-branes present in type IIA (see e.g. ([98–100]) for a more complete discussion). This is indeed the case and we can see that we must have that the M2-brane reduces to the fundamental string F1 when it wraps the M-Theory circle, or to the D2-brane when the worldvolume is transverse to the M-Theory circle S^1_M (see ([101–103]) for a discussion in the case of the BLG model). Similarly, the M5-brane reduces to the D4-brane when wrapping S^1_M (see ([104, 105]) for a more accurate discussion) and to the NS5-brane when its worldvolume is transverse to S^1_M. Obtaining D0 and D6 branes is more complicated, but it turns out that the former correspond to momentum modes along the M-Theory circle ([98]) which indeed needs to be quantised. D6-branes, which are related by electric-magnetic duality to D0-branes (which couple in type IIA to the Ramond-Ramond gauge field A_μ)

$$A_\mu \longrightarrow F_2 \xrightarrow{\star} F_8 \longrightarrow C_7, \tag{4.1.6}$$

lift to Kaluza-Klein monopoles ([98]) in M-Theory, that is supergravity solutions of the form $\mathbb{R}^{1,7} \times \mathcal{TB}$ where $\mathcal{TB}$ is the multi-centered Taub-NUT space.

Therefore from M-Theory we can recover all the D-branes of Type II string theories, by first compactifying on the M-Theory circle and taking the low energy limits of the worldvolume theories on stacks of M2 and M5-branes, and then by possibly T-dualizing to obtain the D-branes of Type IIB string theory ([106]).

4.2 Degrees of Freedom on Parallel M-Branes

Whereas the theories on a stack of D-branes are all related by dimensional reduction and can be obtained from 10-dimensional maximally supersymmetric Yang-Mills Theory, an equally simple description is not available for M-branes. Without a coupling constant, M-Theory allows only a non-perturbative regime and the theories of M2 and M5-branes cannot be obtained from open string scatterings such as in the case of D-branes.

A first puzzle about the M2 and M5 worldvolume theories was related to the degrees of freedom of the two theories. Gauge theories on the worldvolume of N coincident D-branes were known to have a number of degrees of freedom proportional to N^2, which is in accord with the degrees of freedom of $U(N)$ gauge theories or equivalently with the number of Chan-Paton factors for an open string ending on a stack of N D-branes. However, when such degrees of freedom were investigated for the worldvolume theories of N coincident M2 and M5-branes, the degrees of freedom were found to be proportional to $N^{3/2}$ and N^3 respectively ([107]). The origin of such degrees of freedom is still not well understood, but the candidate theories describing such theories should correctly reproduce such scaling of the degrees of freedom (in a sense the N^3 scaling for the M5 theories can be understood from the existence of a BPS state in which the M2-brane has three disconnected boundaries (like a higher dimensional pair of pants) and since M5-branes are supposed to interact through M2-branes the analogy with D-branes and open strings is clear).

In the following we will focus in turn on both the theory describing N coincident M5-branes and N coincident M2-branes. Both theories are supposed to be superconformal field theories (due to the absence of a characteristic length in M-Theory) in 3 and 6 dimensions respectively, and should preserve 16 supercharges, being half BPS object of 11-dimensional supergravity. A considerable progress has been made in the past 10 years in the description of coincident M2-branes. First through the BLG model, which correctly reproduces the dynamics of two M2-branes and successively through the ABJM model describing an arbitrary number of coincident M2-branes in an orbifold background $\mathbb{C}/\mathbb{Z}_k$ (where k appears as the level of the Chern-Simons term of the ABJM model). The theory describing coincident M5-branes, known as the (2,0) theory, has instead remained more elusive, but it nevertheless gave rise to a series of results and dualities between quantum field theories in different dimensions.

4.3 M2-Branes

The BLG model provides a correct description of a pair of M2-branes satisfying all the properties required: superconformal invariance, existence of 16 supercharges and of an $SO(8)$ R-symmetry, $N^{3/2}$ scaling of the degrees of freedom and presence of non-trivial interactions. What came as a surprise was that the gauge symmetry is realized through a so-called 3-algebra rather than through usual Lie algebras. The hint that a different gauge structure was needed to describe parallel M2-branes came from the work of Basu and Harvey ([108]), who first proposed an equation describing coincident M2-branes ending on a single M5-brane.

Recall the similar situation in the case of Dp-D(p+2) branes, where we look at a D1-D3 system as in Table 4.1. The three scalars X^i parametrizing the directions of motion

	0	1	2	3	4	5	6	7	8	9
$D3$	×	×	×	×	·	·	·	·	·	·
$D1$	×	·	·	·	×	·	·	·	·	·

Table 4.1: The D1-D3 system described by Nahm's equation.

of the D1-strings transversal to the D3-brane (since the boundary of the D1-strings is codimension three inside the D3-brane), satisfy the following Nahm equation ([109])

$$\frac{dX^i}{dt} = \frac{1}{2}\varepsilon^{ijk}[X^j, X^k], \tag{4.3.1}$$

where t is a coordinate on the longitudinal direction parametrizing the distance from the D3-brane. As it is well known a solution is provided by a fuzzy sphere S^2, whose radius diverges at the location of the D3-brane, given by ([109–111])

$$X^i = \frac{1}{2t}\sigma^i, \tag{4.3.2}$$

where σ^i are the generators of $SU(2)$ and satisfy the usual commutation relations. Basu and Harvey ([108]) tried to lift such a configuration to M-Theory and describe coincident M2-branes ending on a single M5-brane, as in Table 4.2.

	0	1	2	3	4	5	6	7	8	9	10
$M5$	×	×	×	×	×	×	·	·	·	·	·
$M2$	×	×	·	·	·	·	×	·	·	·	·

Table 4.2: The M2-M5 system described by the Basu-Harvey equation.

Now the four scalars X^a which parametrize the degrees of freedom of the boundary of the M2-branes inside the M5, need to satisfy the following equation ([108])

$$\frac{dX^a}{dt} + \frac{k}{4!}\varepsilon^{abcd}[G, X^b, X^c, X^d] = 0, \tag{4.3.3}$$

where k is a constant and G is a fixed matrix which satisfies $G^2 = \mathbf{1}$. It turns out that a solution is given by a fuzzy S^3, but most importantly we see the appearance of a multi-linear bracket which will be fundamental in the description of the BLG model. In particular the Basu-Harvey equation can be recast in the form

$$\frac{dX^a}{dt} + \frac{1}{4!}\varepsilon^{abcd}[X^b, X^c, X^d] = 0, \tag{4.3.4}$$

where $[\,,\,,\,]$ is the bracket of a 3-algebra. We now turn to the study of such algebraic structures.

4.4 3-Algebras and the BLG Model

Recall that in Super-Yang-Mills theories (on parallel D-branes) a global gauge transformation for a field Φ is given by

$$\delta\Phi = [\alpha, \Phi], \tag{4.4.1}$$

where both α and Φ are matrices in some Lie algebra. In particular the fields were expanded in a vector space with basis $\{T^a\}$ such that $\Phi = \Phi_a T^a$. Then the gauge structure followed from the antisymmetric product

$$[T^a, T^b] = f^{ab}{}_c T^c, \tag{4.4.2}$$

where $f^{ab}{}_c$ are antisymmetric in the upper indices. Then imposing that variations act as derivations

$$\delta([X, Y]) = [\delta X, Y] + [X, \delta Y], \tag{4.4.3}$$

we find the Jacobi identity

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0. \tag{4.4.4}$$

In order to lift the gauge structure to a 3-bracket, we again expand the field in some basis of a vector space $\{T^a\}$, but we define global transformations as

$$\delta\Phi = [\alpha, \beta, \Phi]. \tag{4.4.5}$$

The antisymmetric triple product can be expanded in a basis of the vector space through the structure constants

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d, \tag{4.4.6}$$

which are antisymmetric in the upper indices. Then requiring the derivation property imposes a Jacobi-like identity, known as the fundamental identity

$$[X_1, X_2, [X_3, X_4, X_5]] = [[X_1, X_2, X_3], X_4, X_5] + [X_3, [X_1, X_2, X_4], X_5] + [X_3, X_4, [X_1, X_2, X_5]]. \tag{4.4.7}$$

Therefore we can define a 3-algebra as a vector space endowed with a totally antisymmetric 3-bracket which satisfies the fundamental identity. As for the case of metric Lie algebras, we require the existence of a symmetric inner product

$$h^{ab} = \langle T^a, T^b \rangle. \tag{4.4.8}$$

Requiring invariance of the inner product imposes

$$\langle [W, X, Y], Z \rangle = \langle W, [X, Y, Z] \rangle, \tag{4.4.9}$$

or equivalently the structure constants must be totally antisymmetric when all the indices are raised

$$f^{abcd} = f^{[abcd]}. \tag{4.4.10}$$

It turns out that the existence of a positive definite metric 3-algebra is a very strict requirement, and there exists only one such 3-algebra, see ([112, 113]) for proofs, up to taking direct sums, which is called $\mathcal{A}_4$. It is a 4-dimensional 3-algebra, with four generators T^a, with the 3-bracket defined by

$$f^{abcd} = \frac{2\pi}{k}\varepsilon^{abcd}, \tag{4.4.11}$$

where ε^{abcd} is the totally antisymmetric Levi-Civita tensor and k is a constant.

We can now formulate BLG model ([3, 4]). Let the Lorentz symmetry be broken as

$$SO(1, 10) \to SO(1, 2) \times SO(8)_R \tag{4.4.12}$$

and let X^I $(I = 3, 4, \ldots, 10)$ be eight scalars parametrizing the transverse fluctuations to the branes worldvolume. In order to have supersymmetry we then need eight fermionic degrees of freedom, which are realized from the 32 degrees of freedom of an 11-dimensional Majorana spinor ψ, upon which we impose the following projection condition

$$\Gamma_{012}\epsilon = \epsilon \qquad \Gamma_{012}\psi = -\psi, \tag{4.4.13}$$

(where ϵ is the supersymmetry parameter) which reduces the degrees of freedom to 16. Then on shell we have a match of bosonic and fermionic degrees of freedom. We assume canonical dimensions of the fields in 3 dimensions

$$[X] = 1/2 \quad [\psi] = 1 \qquad [\epsilon] = -1/2. \tag{4.4.14}$$

We promote the derivatives to covariant derivatives through the introduction of a gauge field $(A_\mu)^b_{\ a}$ in the following way

$$D_\mu \Phi_a = \partial_\mu \Phi_a - (A_\mu)^b_{\ a} \Phi_b, \tag{4.4.15}$$

where $\mu = 0, 1, 2$ and Φ is a generic field. As discussed, we already have a match of bosonic and fermionic degrees of freedom, which seems to not allow the introduction of a new gauge field. It will turn out that the gauge field will contribute no additional degrees of freedom and will enter the theory through a Chern-Simons term but without a canonical kinetic term. We can then define the field strength as

$$[D_\mu, D_\nu] \equiv F_{\mu\nu}. \tag{4.4.16}$$

We can now write down the supersymmetry transformations, keeping in mind that since ψ and ϵ have opposite chirality under Γ_{012} we must have an odd number of transverse gamma matrices. The variations of the BLG model are ([3])

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\psi \\ \delta\psi &= \Gamma^\mu \Gamma^I D_\mu X^I \epsilon - \frac{1}{3!}\Gamma^{IJK}[X^I, X^J, X^K]\epsilon \\ \delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_\mu \Gamma^I [X^I, \psi, \cdot], \end{aligned} \tag{4.4.17}$$

where $(\cdot)$ represents an arbitrary field. By requiring closure of the algebra it is found that the 3-bracket needs to satisfy the fundamental identity, meaning that the fields are valued in a 3-algebra. The algebra closes on shell through the following equations of motion for the fields ([3])

$$\begin{aligned} 0 =& D^2 X^I + \frac{1}{2}[[X^I, X^J, X^K], X^J, X^K] + \frac{i}{2}[\bar{\psi}, \Gamma^{IJ}\psi, X^J] \\ 0 =& \Gamma^\alpha D_\alpha \psi + \frac{1}{2}\Gamma^{IJ}[\psi, X^I, X^J] \\ 0 =& F_{\alpha\beta}(\cdot) + \varepsilon_{\alpha\beta\gamma}\left([X^J, D^\gamma X^J, \cdot] + \frac{i}{2}[\bar{\psi}, \Gamma^\gamma \psi, \cdot]\right). \end{aligned} \tag{4.4.18}$$

These equations of motions are invariant under the supersymetry variations (4.4.17). We can derive the equations of motion from the following Lagrangian

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int} + \mathcal{L}_{pot} + \mathcal{L}_{CS}. \tag{4.4.19}$$

We have the kinetic terms for the bosonic and fermonic degrees of freedom

$$\mathcal{L}_{kin} = \frac{1}{2}\langle D_\mu X^I, D^\mu X^I \rangle + \frac{i}{2}\langle \bar{\psi}, \Gamma^\mu D_\mu \psi \rangle, \tag{4.4.20}$$

followed by an interaction term of the form

$$\mathcal{L}_{int} = -\frac{i}{4}\langle[\bar{\psi}, X^I, X^J], \Gamma^{IJ}\psi\rangle, \tag{4.4.21}$$

and a potential term

$$\mathcal{L}_{pot} = \frac{1}{2\cdot 3!}\langle[X^I, X^J, X^K], [X^I, X^J, X^K]\rangle. \tag{4.4.22}$$

Finally the Chern-Simons term can be written as

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd}(A_\mu)_{ab}\partial_\nu(A_\lambda)_{cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}(A_\mu)_{ab}(A_\nu)_{cd}(A_\lambda)_{ef}\right), \tag{4.4.23}$$

where $(A_\mu)^b_{\ a} = f^{cdb}{}_a(A_\mu)_{cd}$. As previously mentioned, there is only one Euclidean 3-algebra, the so-called $\mathcal{A}_4$. Two different groups correspond to this algebra, $SO(4)$ and $Spin(4)$. An interpretation of the BLG model was found by studying the vacua of the theory, which are obtained by minimising the potential, i.e.

$$[X^I, X^J, X^K] = 0. \tag{4.4.24}$$

It turns out that the brane interpretation depends on the level k of the Chern-Simons term. In particular ([114–116]) we have a brane interpretation only in the following cases

$$\begin{cases} k=1 \quad \text{Gauge Group} = SO(4) \quad \longrightarrow 2 \text{ M2-branes in } \mathbb{R}^8 \\ k=2 \quad \text{Gauge Group} = Spin(4) \longrightarrow 2 \text{ M2-branes in } \mathbb{R}^8/\mathbb{Z}_2 \\ k=4 \quad \text{Gauge Group} = SO(4) \quad \longrightarrow 2 \text{ M2-branes in } \mathbb{R}^8/\mathbb{Z}_2, \end{cases} \tag{4.4.25}$$

where in the last case a discrete torsion is present for the background four-form.

4.5 M5-Branes

The possible existence of a 6-dimensional superconformal theory first followed from Nahm's classification ([117]) of superconformal algebras, which showed that they existed only for space-time dimensions equal to and lower than six. In particular two possible 6-dimensional superconformal algebras were shown to exist, the so-called $(1,0)$ and (2,0) algebras. They differ in the amount of supersymmetry, 8 supercharges and 16 supercharges (the maximal amount in 6 dimensions) respectively, and on the R-symmetry group, $SU(2)$ for the (1,0) theory and $SO(5)$ for the (2,0) theory.

From now on we will only focus on the latter, but interestingly progress has been made recently on the former through F-Theoretic methods ([118]).

The existence of the (2,0) algebra in itself does not show that there must be an actual field theoretic representation on 6-dimensional fields, or at least it does not demand the existence of a field theory in the way we are used to think about it. Concrete evidence for the existence of such a theory was brought forward by Witten ([119]) a few years later. In particular, Witten looked at type IIB string theory on a space of the form $\mathbb{R}^{1,3} \times K3$ at particular points of the moduli space of K3 surfaces. It is known that at such points, a set of two-spheres in the K3 surfaces shrinks to zero size and they follow an ADE classification depending on the number and the intersections of the two-spheres in the K3 surface. In ([119]) it was argued that upon compactification of type IIB string theory on K3 surfaces developing singularities at these points of the moduli space, a superconformal field theory in 6-dimensions could be realized. Nevertheless, it was realized that it could not have a simple perturbative description in terms of string states, since by wrapping D3-branes along the two cycles of the K3 surface, strings were obtained whose tension was proportional to the area of the cycles themselves. Therefore in the singular limit in which the (2,0) theory was realized and the two-spheres shrank to zero size, such strings would become tension-less. The (2,0) theories, which as we saw admit an ADE classification, still lack a satisfying description, but they have nevertheless been useful for numerous results and dualities for lower dimensional supersymmetric theories related by a web of compactifications.

Following the work of Nahm ([117]), the (2,0) supersymmetry can be realized on an abelian tensor multiplet consisting of five scalars X^I, a self dual three-form $H_{\mu\nu\lambda}$ and the fermionic super-partners Ψ. The fields tranform under the Lorentz group $SO(1,5)$ and the R-symmetry group $SO(5)_R \simeq Sp(4)_R$. Moreover we can take the dimensions of the fields to be

$$[X] = 2 \ , \qquad [\Psi] = 5/2 \ , \qquad [H] = 3. \tag{4.5.1}$$

This tensor multiplet describes a single M5-brane, where the five scalars parametrize the transverse fluctuations to the brane in 11-dimensional space-time. We can write down the supersymmetry variations of the fields in the following form ([6])

$$\begin{aligned} \delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\ \delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi \\ \delta\Psi &= \Gamma^\mu\Gamma^i D_\mu X^i\epsilon + \frac{1}{2\cdot 3!}H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon, \end{aligned} \tag{4.5.2}$$

where $\mu, \nu = 0, 1, \ldots, 5$ and $i, j = 6, 7, \ldots, 10$. The gamma matrices are 32×32 matrix representations of the Clifford algebra in 11 dimensions and the H field is self dual

$$H_{\mu\nu\lambda} = \frac{1}{6}\varepsilon_{\mu\nu\lambda\sigma\tau\omega}H^{\sigma\tau\omega}. \tag{4.5.3}$$

The fermionic degrees of freedom Ψ and the supersymmetry parameter ϵ are 32 components and satisfy the following chirality conditions

$$\Gamma_{012345}\epsilon = \epsilon \qquad \Gamma_{012345}\Psi = -\Psi. \tag{4.5.4}$$

The following equations of motion are invariant under the (2,0) supersymmetry described by the variations (4.5.2)

$$\begin{gathered} D^2X^i = 0 \\ \Gamma^\mu D_\mu \Psi = 0 \\ H = \star H \qquad dH = 0. \end{gathered} \tag{4.5.5}$$

Note that we could write a Lagrangian for the scalar fields X^i

$$S_{X^i} = \int d^6x \; D_\mu X^i D^\mu X^i \tag{4.5.6}$$

and for the Fermionic degrees of freedom

$$S_\Psi = \int d^6x \; \bar{\Psi}\Gamma^\mu D_\mu \Psi. \tag{4.5.7}$$

Nevertheless we see that the usual action that we would write down for the H field

$$S_H = \int_{\mathbb{R}^{1,5}} H \wedge \star H, \tag{4.5.8}$$

is of no use since H is self-dual and $H \wedge \star H = H \wedge H = 0$. A similar inconsistency in deriving an action, pointed out in ([120]), is that if we consider the (2,0) theory on a product manifold $M_5 \times S^1$ we see that in the low energy limit the theory on M_5 should be proportional to R^{-1} (where R is the radius of the circle S^1) due to conformal invariance in 6 dimensions, but at the same time it is clear that integrating over the circle direction would give a factor of R in the low energy theory on M_5 coming from $\sqrt{g}$ in the action. Nevertheless actions for a single M5-brane have been written down, see ([121–126]). We now turn to the possible application of 3-algebras to the study of parallel M5-branes.

4.6 3-Algebras and M5-branes

In ([6]) the authors proposed a set of equations of motion for a non-abelian tensor multiplet in 6 dimensions invariant under (2,0) supersymmetry. In a similar fashion to the construction of the BLG model, the authors proposed a non-abelian extension of the free equations of motion, such that the fields are required to live in a generic vector space endowed with an antisymmetric triple product. Closure of the supersymmetry algebra then requires that the vector space be actually a 3-algebra, or equivalently that the triple bracket should satisfy the fundamental identity (4.4.7). We will now look at the specifics of such equations of motions and how the gauge symmetry is realized through 3-algebras rather than usual Lie algebras. As in the BLG model, the introduction of a gauge field covariantize the derivatives ([6])

$$D_\mu \Phi_a = \partial_\mu \Phi_a - (A_\mu)_a^b \Phi_b, \tag{4.6.1}$$

where Φ is a generic field which has been expanded in a basis $\{T^a\}$ of the 3-algebra as $\Phi = \Phi_a T^a$. A fundamental difference to the BLG model is the necessary introduction of a new field, a vector Y^μ which will turn out not to transform under supersymmetry. It is then possible to express the supersymmetry transformations which realize the (2,0) algebra as follows, where we state again that the 3-algebra structure actually follows from closure of the algebra on the non-abelian tensor multiplet ([6])

$$\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i \Psi \\
\delta Y^\mu &= 0 \\
\delta \Psi &= \Gamma^\mu \Gamma^i D_\mu X^i \epsilon + \frac{1}{2 \cdot 3!} H_{\mu\nu\lambda} \Gamma^{\mu\nu\lambda} \epsilon - \frac{1}{2} \Gamma_\mu \Gamma^{ij} [Y^\mu, X^i, X^j] \epsilon \\
\delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu} D_{\lambda]} \Psi + i\bar{\epsilon}\Gamma^i \Gamma_{\mu\nu\lambda\rho} [Y^\rho, X^i, \Psi] \\
\delta A_\mu(\cdot) &= i\bar{\epsilon}\Gamma_{\mu\nu} [Y^\nu, \Psi, \ \cdot\ ].
\end{aligned} \tag{4.6.2}$$

The conventions are the same of the previous section, where we presented the free tensor multiplet. The Fermionic degrees of freedom satisfy the chirality conditions (4.5.4) and a dot $(\cdot)$ represents an arbitrary field. Note that the gauge field can be taken to have canonical dimension $[A_\mu] = 1$, and it follows that the vector field Y^μ has dimensions $[Y^\mu] = -1$. These supersymmetry transformations close on the tensor multiplet if the following equations of motion and constraints are satisfied ([6])

$$\begin{aligned}
0 = &\quad D^2 X^i - \frac{i}{2}[Y^\mu, \bar{\Psi}, \Gamma_\mu \Gamma^i \Psi] - [Y^\mu, X^j, [Y_\mu, X^j, X^i]] \\
0 = &\quad D_{[\mu} H_{\nu\lambda\rho} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar{\Psi}, \Gamma^\tau \Psi] \\
0 = &\quad \Gamma^\mu D_\mu \Psi + \Gamma^\mu \Gamma^i [Y_\mu, X^i, \Psi] \\
0 = &\quad F_{\mu\nu}(\cdot) - [Y^\lambda, H_{\mu\nu\lambda}, \ \cdot\ ] \\
0 = &\quad D_\mu Y^\nu = [Y^\mu, Y^\nu, \cdot] = [Y^\mu, D_\mu(\cdot), \cdot']. \qquad (4.6.3)
\end{aligned}$$

Note that the constraint $F_{\mu\nu}(\cdot) = [Y^\lambda, H_{\mu\nu\lambda}, \ \cdot\ ]$ can be interpreted as the vanishing of the so-called *fake curvature* in higher gauge theories. In that context, in order to have a consistent parallel transport of 1-dimensional objects, such quantity needs to vanish (see ([91,127]) for an interpretation of the Lambert-Papageorgakis model discussing such aspects). Moreover it was shown in ([6]) that it is not possible to consistently define a potential $B_{\mu\nu}$ for the field strength $H_{\mu\nu\lambda}$. Since the vector Y^μ is covariantly constant, we can single out a direction in space-time and in the gauge algebra and study the set of equations around a particular expectation value.

Recall that a Lorentzian 3-algebra can be constructed starting from a Lie algebra $\mathcal{G}$ by adding two generators $T^\pm$ and defining the structure constants of the 3-algebra as

$$f^{+ab}{}_c = f^{ab}{}_c \quad f^{abc}{}_- = f^{abc}, \qquad (4.6.4)$$

where f^{abc} are the structure constants of the Lie algebra. It turns out ([6]), that for a Lorentzian 3-algebra, expanding around a particular value of Y^μ

$$\langle Y^\mu_a \rangle = g\, \delta^\mu_5 \delta^+_a \qquad (4.6.5)$$

reduces the equations (4.6.3) to those of 5-dimensional Super-Yang-Mills (SYM) theory (and two abelian free tensor multiplet). Similarly, for the Euclidean 3-algebra $\mathcal{A}_4$, expanding around a particular value of Y^μ

$$\langle Y^\mu_a \rangle = g\, \delta^\mu_5 \delta^4_a \qquad (4.6.6)$$

reproduces 5-dimensional Super-Yang-Mills theory and only one abelian free multiplet. In fact, one could have engineered the algebra (4.6.2) starting from 5-dimensional Super-Yang-Mills theory by suitably identifying the fields, in a similar fashion as it was done for the case of M2-branes in ([128]). One could then argue that the equations of motion here presented are just a reformulation of the theory on D4-branes in a language of 3-algebras and where conformal invariance is manifest. However the analysis carried out in ([129]) seems to confirm another proposal for the dynamics on coincident M5-branes. The authors give a null expectation value to the vector Y^μ and show that the system of equations reduces to motion on the instanton moduli space. By quantising such system the proposal of ([130,131]) for a light-cone description of the (2,0) theory is recovered, thus suggesting that the 6-dimensional (2,0) algebra of ([6]) is more than just a reformulation of the theory on D4-branes. Indeed, depending on the expectation value of Y^μ, it is possible to reduce the (2,0) algebra to either 5-dimensional Super-Yang-Mills theory or to quantum mechanics on the instanton moduli space, and both of these theories are known to capture some aspects of the dynamics on parallel M5 branes. In this thesis we try to bring forward more evidence for such proposal by extending it and relating it to the BLG model describing two M2-branes.

In Chapter 5 we extend the construction of ([6]). By introducing an abelian three-form $C_{\mu\nu\lambda}$ into the algebra, we find an extended representation on a 6- dimensional non-abelian tensor multiplet. Requiring closure of the algebra on the tensor multiplet provide new equations of motion and constraints for the fields, which reduce to (4.6.3) in the case in which the three-form is switched off. In the case in which $C_{\mu\nu\lambda} \neq 0$, solving the constraints for the fields naturally makes a reduction to the BLG model in 3 dimensions manifest. Therefore the extended (2,0) algebra that we propose reduces to the description of two coincident M2-branes when the three-form is switched on, and thus corroborates the proposal for a description of parallel M5-branes of ([6]).

4.7 M5-Branes and Dualities

Even though the exact formulation of the (2,0) theory is not clear in 6 dimensions, it is possible to obtain a number of results in lower dimensional field theory by making use of the few facts which are known about the (2,0) theory. In particular, thanks to it, it is possible to find a web of unexpected dualities between quantum field theories in different dimensions, which can receive an interpretation thanks to different compactification limits of the (2,0) theory. In this section we are going to see a few examples in which lower dimensional quantum field theories are interpreted as such compactifications, and we will see that specific quantities in such theories can then be interpreted as geometric properties of the compactification manifold. We then proceed to discuss how different compactifications of the (2,0) theory can be related to produce dualities thanks to the conformal invariance of the (2,0) theory.

The main result needed to understand such dualities follows from the reduction of the (2,0) theory on a circle. It is found that in the limit in which the radius of the circle vanishes, we can recover 5-dimensional $\mathcal{N}=2$ Super-Yang-Mills theory. Such theory has the maximal amount of supersymmetry in 5-dimensions, and a field content consisting of 5 scalars X^i, a gauge field A_μ (with field strength $F_{\mu\nu}$) and fermionic super-partners Ψ. It is possible to write down an action, which reads

$$\begin{aligned} S_{SYM}^{5d} = -\frac{1}{g_{5d}^2}\int d^5x\, \Big(\frac{1}{4}F_{\dot{\mu}\dot{\nu}}F^{\dot{\mu}\dot{\nu}} + \frac{1}{2}D_{\dot{\mu}}X^iD^{\dot{\mu}}X^i - \frac{i}{2}\bar{\Psi}\Gamma^{\dot{\mu}}D_{\dot{\mu}}\Psi \\ + \frac{1}{2}\bar{\Psi}\Gamma^5\Gamma^i[X^i,\Psi] - \frac{1}{4}\sum_{i,j}[X^i,X^j]^2\Big), \end{aligned} \tag{4.7.1}$$

where $\dot{\mu},\dot{\nu}=0,1,\ldots,4$ and $i,j=6,7,\ldots,10$. In particular we see that the reduction of the self-dual $H_{\mu\nu\lambda}$ field to the field strength $F_{\mu\nu}$ allows a usual gauge theoretic description of the theory in contrast to what happened in 6-dimensions. Such theory is otherwise obtained by dimensional reduction of the $\mathcal{N}=1$, 10-dimensional Super-Yang-Mills theory. Such reduction can be interpreted in the string theory context, by recalling that in going from M-Theory to type IIA string theory, M5-branes wrapping the M-Theory circle are understood to be described by parallel D4-branes, whose low-energy theory is indeed 5-dimensional Super-Yang-Mills theory. We refer to ([104]) for a more thorough discussion.

The coupling constant of 5-dimensional Super-Yang-Mills theory is given, in terms of the radius R of the circle on which we compactify the (2,0) theory, by

$$g_{5d}^2 = R. \tag{4.7.2}$$

Therefore we can consider the (2,0) theory as the UV fixed point of the theory on coincident D4-branes. We see a relation similar to the one defining the string coupling in Type IIA string theory and the radius of the M-Theory circle. In the limit in which the radius goes to zero we gain a perturbative understanding in terms of 5-dimensional Super-Yang Mills theory, but at strong coupling a new dimension opens up. 5-dimensional Super-Yang-Mills theory is naively power counting non-renormalizable, implying that a quantum theory is not well defined without additional degrees of freedom. In fact it was argued ([104, 105]) that 5-dimensional Super-Yang-Mills theory contains all the degrees of freedom of the (2,0) theory on a circle S^1 and in particular that the instantons of the 5-dimensional theory are exactly Kaluza-Klein states coming from the (2,0) theory. Indeed, instantons in 5-dimensional Super-Yang-Mills theory are string-like uplifts of the usual magnetic monopoles of 4-dimensional theories which can be thought of as the strings arising from M2-branes ending on M5-branes. This was argued to be the case by matching the super-algebras $\{Q_\alpha, Q_\beta\}$ of the (2,0) theory on S^1 and of 5-dimensional Super-Yang-Mills theory ([104]).

Once we understand the statement that the (2,0) theory on a circle reduces to 5-dimensional $\mathcal{N} = 2$ Super-Yang-Mills theory, we can start to gain insight into lower dimensional compactifications, and in particular into the geometric interpretations which arise in such cases. Reducing further on a second circle, following what we said so far, will produce 4-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory. This statement is well understood both in the reduction of maximally supersymmetric Super-Yang-Mills theories and in the parallel brane picture where D4-branes are known to reduce to D3-branes when wrapping a vanishing circle. This is the simplest set up in which the 6-dimensional (2,0) theory can provide us with valuable geometric interpretations. In particular, recall that the the coupling constant e and the θ parameter of $\mathcal{N} = 4$ SYM are usually coupled as

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}, \tag{4.7.3}$$

in order to study S-duality transformations. In particular 4-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory is invariant under shift $\tau \to \tau + 1$ and is conjectured ([132]) to be invariant under S-duality transformations sending $\tau \to -\frac{1}{\tau}$. What the (2,0) theory affords is to think the S-duality group generated by such transformations as modular transformations of the complex structure of the torus that was used to reduce form 6 to 4 dimensions to obtain the 4-dimensional Super-Yang-Mills theory. It follows that the invariance under transformations which send $\tau \to \frac{a\tau+b}{c\tau+d}$ is nothing but the modular invariance of the complex structure of the torus upon which we reduce the (2,0) theory (see section 2.2 for a full discussion). Reduction of the (2,0) theory on different and higher dimensional manifolds leads to a number of dualities and results which we will now briefly survey.

An important breakthrough in this direction was realized by Gaiotto ([133]) with the engineering of a whole new class of superconformal theories in 4 dimensions with 16 supercharges. These theories, called theories of class $\mathcal{S}$[1], were realized as the low energy limit of the (2,0) theory on a Riemann surface of genus g $\Sigma_{n,g}$ with possibly a number of punctures. These are point-like defects at which the fields develop prescribed singularities. We will see again how thinking of such 4-dimensional theories as obtained from the 6-dimensional (2,0) theory allows us to interpret important quantities as geometric properties of the compactification manifolds.

In particular the (2,0) theory associated to the lie algebra $\mathfrak{g}$, denoted $T[\mathfrak{g}]$, admits a brane interpretation in M-Theory for the two series A_n and D_n (while the E_6,E_7 and E_8 theories do not admit such a description). The A_n series represents $(n+1)$ parallel M5-branes, while the D_n series is associated to the description of $2n$ M5-branes on top of an orientifold singularity obtained by an action on the five transverse directions to the worldvolume of the branes. Class $\mathcal{S}$ theories for the A_n series are obtained by topologically twisting the (2,0) theory on a Riemann surface $\Sigma_{n,g}$. The topological twisting carried out allows to preserve supersymmetry on a non-flat space which would otherwise allow no covariantly constant spinors, needed for supersymmetry. In particular, in 6 dimensions we have a Lorentz group $SO(1,5)$ and an R-symmetry group $Sp(4)_r \simeq SO(5)_R$. When we consider the (2,0) theory on a product space of the form $M_4 \times \Sigma_{n,g}$, as it is the case for class $\mathcal{S}$ theories, these symmetry groups break to

$$\begin{aligned} SO(1,5) &\longrightarrow SO(1,3) \times SO(2)_H \\ SO(5)_R &\longrightarrow SO(3)_R \times SO(2)_R, \end{aligned} \tag{4.7.4}$$

where $SO(2)_H$ represents the holonomy group of the 2-dimensional manifold $\Sigma_{n,g}$. Such an holonomy represents exactly the obstacle to have covariantly constant spinors, since we see that

$$D_\mu \epsilon = (\partial_\mu + \omega_\mu)\ \epsilon, \tag{4.7.5}$$

where ω_μ is a non-trivial spin connection contributing to the covariant derivative D_μ. In order to solve this problem, we carry out a topological twist, that is, we look for a way to get rid of the unwanted spin connection. The way to do this is to redefine the

[1] $\mathcal{S}$ as in six.

Lorentz group on $\Sigma_{n,g}$ in such a way that the spinor parameter does not transform in a Fermion representation any more, but as a scalar (which will allow to take it to be covariantly constant). In particular the supercharges of the (2,0) theory transform in the $(\mathbf{4}\otimes\mathbf{4})$ representation of $SO(1,5)\oplus SO(5)_R$. Once we break the symmetry groups, the representation decomposes to ([133])

$$\left((\mathbf{2},\mathbf{1})_{\frac{1}{2}}\oplus(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}\otimes(\mathbf{2}_{\frac{1}{2}}\oplus\mathbf{2}_{-\frac{1}{2}})\right). \tag{4.7.6}$$

So as to have supercharges which transform as scalars on $\Sigma_{n,g}$, we redefine the Lorentz group as

$$SO(2)' = SO(2)\times SO(2)_R|_{\text{diag}}, \tag{4.7.7}$$

that is we take the diagonal subgroup of $SO(2)\times SO(2)_R$ (which can be done by either adding or subtracting the $U(1)$ charges). As a consequence the supercharges now transform as

$$(\mathbf{2},\mathbf{1},\mathbf{2})_1\oplus(\mathbf{2},\mathbf{1},\mathbf{2})_0\oplus(\mathbf{2},\mathbf{2},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_{-1}, \tag{4.7.8}$$

and we see that half of the supercharges now transform as scalars on $\Sigma_{n,g}$. Therefore 16 supercharges are preserved in the 4-dimensional theory, thus giving $\mathcal{N}=2$ supersymmetry. Indeed, we see that the symmetry group of the 4-dimensional theory obtained by shrinking $\Sigma_{n,g}$ to zero size is

$$SO(1,3)\times SU(2)_R, \tag{4.7.9}$$

which agrees with the symmetry group of $\mathcal{N}=2$ supersymmetric gauge theories in 4 dimensions. In fact, theories of class $\mathcal{S}$ are actually super-conformal. What effectively happened is that we introduced an R-symmetry gauge field A_μ which corrects the covariant derivative on the spinor

$$D_\mu\epsilon = (\partial_\mu+\omega_\mu+A_\mu)\,\epsilon, \tag{4.7.10}$$

and which can be tuned to cancel the contribution from the spin connection, thus allowing for covariantly constant spinors. It turns out that through Gaiotto's construction the Seiberg-Witten curve for the theories of class $\mathcal{S}$ are nothing other than n-sheeted covers of the Riemann surfaces $\Sigma_{n,g}$ that were used to reduce the (2,0) theory to 4 dimensions.

The geometries of the compactification manifolds give insight into the 4-dimensional theory, as the space of exactly marginal deformations (preserving supersymmetry and conformal symmetry) can be identified with the complex structure moduli space of Riemann surfaces $\Sigma_{n,g}$. A whole dictionary was worked out translating between gauge theory

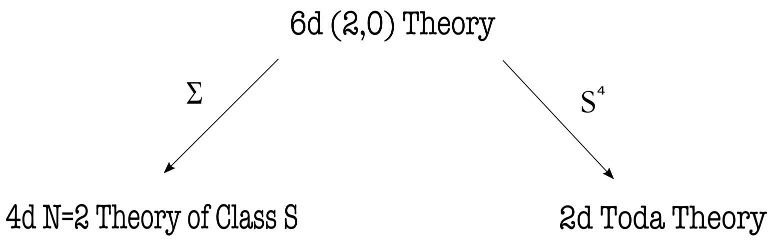


Figure 4.1: The two compactifications giving rise to the AGT correspondence. Equating the partition functions allows the non-trivial identification of quantities in the two different theories.

aspects such as couplings, matter and global symmetries on one side and operations on the geometries such as cutting and gluing on the other side. Importantly, many of the theories of class $\mathcal{S}$ do not admit a Lagrangian description. Drawing from the work of Gaiotto, Alday, Tachikawa and Gaiotto himself ([134]), went on to discover a surprising duality between theories of class $\mathcal{S}$ and 2-dimensional Toda theories, where the latter are non-supersymmetric conformal field theories which also admit an ADE classification. In particular the following set up was used, see Figure 4.1. The (2,0) theory is compactified first on a Riemann Surface $\Sigma_{n,g}$ and the partition function is calculated on S^4 by making use that the latter is conformally flat. Or the order of compactification is reversed thanks to the factorization of the partition function

$$Z[\Sigma_{n,g} \times S^4] = Z[\Sigma_{n,g}] \times Z[S^4]. \tag{4.7.11}$$

Note that the conformal invariance of the (2,0) theory allows to scale either side of the compactifiction and we find that

$$\mathcal{Z}[S^4] = \mathcal{Z}[\Sigma], \tag{4.7.12}$$

where

$$\mathcal{Z}[S^4] \equiv Z[\Sigma_{n,g} \times S^4]|_{\mathrm{Vol}(\Sigma)\to 0} \qquad \mathcal{Z}[\Sigma] \equiv Z[\Sigma_{n,g} \times S^4]|_{\mathrm{Vol}(S^4)\to 0}. \tag{4.7.13}$$

We can then calculate the partition function of the Toda theory of type $\mathfrak{g}$, where $\mathfrak{g}$ is the algebra associated to the initial 6-dimensional (2,0) theory. By equating the two partition functions, of the 4-dimensional theory on S^4, and of the 2-dimensional theory on $\Sigma_{n,g}$, the AGT correspondence allows to identify non-trivial quantity in the two different theories.

Following the AGT conjecture a number of dualities were proposed in the same spirit in order to relate different compactifications of the (2,0) theory. In ([135]) the author

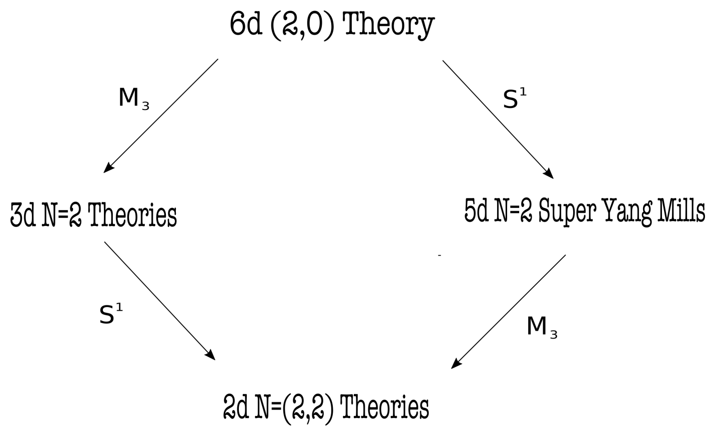


Figure 4.2: The two compactifications giving rise to the 3d-3d correspondence. In particular, via 5-dimensional Super-Yang-Mills theory the moduli space of supersymmetric vacua of $T[M_3]$ is identified with the space of complex flat connections on M_3.

proposed for the first time a 3d-3d correspondence, relating 3-dimensional $\mathcal{N}=2$ theories and complex Chern-Simons theory on a three-manifold M_3. The 3-dimensional supersymmetric theories were obtained again from compactifying the (2,0) theory on M_3 and topologically twisting away the $SO(3)$ holonomy of the manifold in order to obtain a supersymmetric theory. It was argued that when such theories were studied on a space-time of the form $\mathbb{R}^2 \times S^1$, as in Figure 4.2, the moduli space of supersymmetric vacua could be identified with the space of complex flat connections on M_3, that is ([136])

$$\mathcal{M}_{\text{SUSY}}(T[M_3, G]) = \mathcal{M}_{\text{flat}}(M_3, G_{\mathbb{C}}) \tag{4.7.14}$$

where $T[M_3, G]$ is the 3-dimensional $\mathcal{N}=2$ theory obtained by compactifying the (2,0) theory of type $\mathfrak{g} = \text{Lie}(G)$ on M_3. This can be deduced by reversing the order of compactification ([137]) through the known reduction of the (2,0) theory on S^1 to 5-dimensional Super-Yang-Mills theory.

Similarly, in ([138]), the authors proposed a correspondence, as depicted in Figure 4.3, between 2d $\mathcal{N}=(2,0)$ theories labelled by four-manifolds (that is, obtained by compactifying the (2,0) theory on a four-manifold) and defined on a two-torus T^2, with 4-dimensional $\mathcal{N}=4$ Super-Yang-Mills theory (obtained by compactifying the (2,0) theory on T^2) defined on the four-manifold. In order to preserve supersymmetry, a topological twist is necessary on the side of the compactification which leads to the 2d theory since a generic four-manifold has an $SO(4)$ holonomy which does not admit covariantly constant spinors. On the other hand the reduction on the flat two-torus does not need any twisting.

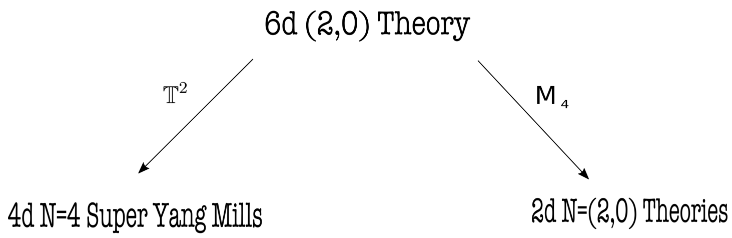


Figure 4.3: The two compactifications giving rise to the 2d-4d correspondence. The elliptic genus of the 2d theory can then be identified with the partition function of the 4-dimensional theory after performing a Vafa-Witten twist.

Then by identifying the partition functions an equality was proposed between the elliptic genus of the 2d theory and the partition function of 4-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory as defined by the Vafa-Witten twist, ([139]).

In Chapter 6, we will extend this line of ideas, by reducing the (2,0) theory on a two-sphere S^2. This analysis was not included in the AGT construction, as the theory thus obtained is not a theory of class $\mathcal{S}$ and in particular is not conformal ([140]). Note that since the two-sphere is not flat, a topological twisting is necessary to allow for covariantly constant spinors, or equivalently, an R-symmetry gauge field needs to be turned on to cancel the spin connection arising from the curvature of the two-sphere.

Chapter 5

M2-Branes And The (2,0) Superalgebra

Dp-branes are all related to each other in a straightforward way using T-duality which is valid microscopically in the open string description and also is manifest in the low energy Yang-Mills effective actions ([141]), although of course the quantum behaviour of these theories drastically depends on their dimension. Mathematically this occurs because all Super-Yang-Mills theories on D-branes are constructed by dimensional reduction of the 10-dimensional Super-Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry.

While the field theories for multiple M2-branes are now known ([4, 5, 142]) (for a review see ([143])) the M5-brane remains mysterious and believed to be non-lagrangian. Although there are various proposals for M5-brane dynamics that use a lagrangian but which require some specific limit to be taken ([104,105,130,144]). One still expects there to be some form of T-duality, inherited from string theory, that relates M5-branes to M2-branes. Even though there is no microscopic picture of these theories analogous to open strings one may still expect to see some universal structure in their field theory descriptions.

One attempt to relate the M2-branes to M5-branes using T-duality was given in ([145]). The simple translational orbifold approach used in ([141]) fails as translations are not a symmetry of the M2-brane Lagrangian. Nevertheless the modified approach of ([145]) leads from the periodic array of M2-branes to a variation of 5-dimensional Super-Yang-Mills as a description of M5-branes.

There have also been papers which show that maximally supersymmetric M2-brane Lagrangian with a Nambu bracket for the 3-algebra leads to an abelian M5-brane ([146–

148]). It might be possible to view the results here in a complimentary context: starting from the non-abelian $(2,0)$ superalgebra associated to multiple M5-branes and then obtaining M2-branes.

In this chapter we will generalise the 6-dimensional (2,0) superalgebra construction of ([6]) by including a non-dynamical abelian background three-form.[1] Setting this to zero reproduces the previous results which have been proposed as a description of two M5-branes (here we specialise to the case of a positive definite Lie-3-algebra). In particular there is a covariantly constant vector which imposes constraints that require there to be an isometry along one direction which leads to 5-dimensional super-Yang-Mills in the spacelike case ([6]), 5-dimensional euclidean Super-Yang-Mills in the timelike case ([144]) and quantum mechanics on instanton moduli space in the null case ([129]). These have all been argued to provide a description of the quantum $(2,0)$ theory ([104, 105, 130, 144]). We then show that turning on the background three-form allows some components of the vector to be dynamical but also forces a dimensional reduction to 3 dimensions leading to the maximally supersymmetric field theory of two M2-branes ([4, 142]). Thus this generalized $(2,0)$ superalgebra provides a structure that contains aspects of both multiple M2-branes and M5-branes.

The structure of the chapter is as follows. In section 5.1 we propose a generalization of the algebra through the introduction of an abelian three-form $C_{\mu\nu\lambda}$, close the algebra and derive the constraints and equations of motion for the fields. In section 5.2 we find the central charges and the energy-momentum tensor associated to the generalized (2,0) algebra. In section 5.3 we relate our construction to the maximally supersymmetric model describing two M2-branes and carry out the reduction.

5.1 Closure of the Algebra

Recall the discussion in section 4.6. In ([6]) a (2,0) algebra was realised on a non-abelian 6-dimensional tensor multiplet. In order to realise the algebra, it was necessary to require the existence of a gauge field A_μ and a spacetime vector Y^μ. The fields were assumed to live in a generic vector space endowed with an antisymmetric triple bracket; similarly to what happens for the BLG model, closure of the algebra required the fields to actually take values in a 3-algebra. Closure also determined the equations of motion for the fields of the tensor multiplet and constraints for the additional fields. The aim of this chapter is to generalise this algebra by including an abelian non-dynamical three-form $C_{\mu\nu\lambda}$ with mass dimension $[C] = -3$.

[1] Using such a three-form has also been considered by A. Gustavsson ([149]).

We consider the following extension of the (2,0) algebra

$$\begin{aligned}
\delta X^i =& i\bar{\epsilon}\Gamma^i\Psi \\
\delta Y^\mu =& \frac{i\alpha}{3!}\bar{\epsilon}\Gamma_{\lambda\rho}C^{\mu\lambda\rho}\Psi \\
\delta\Psi =& \Gamma^\mu\Gamma^i D_\mu X^i\epsilon + \frac{1}{2\cdot 3!}H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon \\
& - \frac{1}{2}\Gamma_\mu\Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon + \frac{\beta}{3!}C_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\Gamma^{ijk}[X^i, X^j, X^k]\epsilon \\
\delta H_{\mu\nu\lambda} =& 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi + i\bar{\epsilon}\Gamma^i\Gamma_{\mu\nu\lambda\rho}[Y^\rho, X^i, \Psi] \\
& + i\gamma\bar{\epsilon}(\star C)_{\mu\nu\lambda}\Gamma^{ij}[X^i, X^j, \Psi] + \frac{i\gamma'}{2}\bar{\epsilon}\Gamma_{[\mu\nu|\rho\sigma}C^{\rho\sigma}{}_{\lambda]}\Gamma^{ij}[X^i, X^j, \Psi] \\
\delta A_\mu(\cdot) =& i\bar{\epsilon}\Gamma_{\mu\nu}[Y^\nu, \Psi, \ \cdot\ ] + \frac{i\delta}{3!}\bar{\epsilon}C^{\nu\lambda\rho}\Gamma_{\mu\lambda\rho}\Gamma^i[X^i, \Psi, \ \cdot\ ] \ , \qquad (5.1.1)
\end{aligned}$$

where $\alpha, \beta, \gamma, \gamma', \delta$ are constants to be determined and a dot $(\cdot)$ denotes an arbitrary field. There are additional terms that one could consider however the rationale behind this choice of algebra will become clear upon showing how a natural reduction to the M2-branes arises. In this section we will show that the superalgebra closes on shell and we will derive the equations of motion and the constraints that the fields need to satisfy.

Before we consider the closure of the algebra we first observe that the fermion equation of motion can be obtained by imposing self-duality of δH. We find that

$$\delta H_{\mu\nu\lambda} - (\star\delta H)_{\mu\nu\lambda} = i\bar{\epsilon}\Gamma_{\mu\nu\lambda}(\Gamma^\rho D_\rho\Psi + \Gamma_\rho\Gamma^i[Y^\rho, X^i, \Psi] + \frac{\gamma}{3!}\Gamma_{\rho\sigma\tau}C^{\rho\sigma\tau}\Gamma^{ij}[X^i, X^j, \Psi]) \ , \qquad (5.1.2)$$

provided that $\gamma' = 3\gamma$ (otherwise one does not find a single expression on the right hand side). Thus we see that the Fermion equation of motion is

$$\Gamma^\rho D_\rho\Psi + \Gamma_\rho\Gamma^i[Y^\rho, X^i, \Psi] + \frac{\gamma}{3!}\Gamma_{\rho\sigma\tau}C^{\rho\sigma\tau}\Gamma^{ij}[X^i, X^j, \Psi] = 0 \ . \qquad (5.1.3)$$

5.1.1 Closure on X^i

We now proceed to close the algebra on the scalar fields X^i. We see that the algebra closes up to a translation and a gauge transformation, that is

$$[\delta_1, \delta_2]X^i = v^\nu D_\nu X^i + \Lambda(X^i) \ , \qquad (5.1.4)$$

with

$$\begin{aligned}
v^\mu &= -2i(\bar{\epsilon}_2\Gamma^\mu\epsilon_1) \\
\Lambda(\cdot) &= -2i(\bar{\epsilon}_2\Gamma_\lambda\Gamma^i\epsilon_1)[Y^\lambda, X^i, \ \cdot\ ] - i\beta(\bar{\epsilon}_2\Gamma_{\mu\nu\lambda}\Gamma^{jk}\epsilon_1)C^{\mu\nu\lambda}[X^j, X^k, \ \cdot\ ] \ . \qquad (5.1.5)
\end{aligned}$$

We note that a new term, proportional to $C_{\mu\nu\lambda}$, now contributes to the definition of gauge transformation compared to the one defined in ([6]).

5.1.2 Closure on Y^μ

Next we look at closing supersymmetry on Y^μ. The expected form of the closure is

$$[\delta_1, \delta_2]Y^\mu = v^\nu D_\nu Y^\mu + \Lambda(Y^\mu) \ , \tag{5.1.6}$$

with v^μ and $\Lambda(\cdot)$ as defined in (5.1.5). Explicit calculation leads to

$$\begin{aligned}[\delta_1, \delta_2]Y^\mu = & -\frac{i\alpha}{3}(\bar{\epsilon}_2\Gamma^\nu\epsilon_1)\ C^{\mu\lambda\rho}H_{\nu\lambda\rho} + \frac{2i\alpha}{3}\left(\bar{\epsilon}_2\Gamma_\nu\Gamma^i\epsilon_1\right)\ C^{\mu\nu\sigma}D_\sigma X^i \\ & -\frac{i\alpha}{6}\left(\bar{\epsilon}_2\Gamma_{\lambda\rho\sigma}\Gamma^{ij}\epsilon_1\right)C^{\mu\lambda\rho}[Y^\sigma, X^i, X^j] \\ & +\frac{i\alpha\beta}{3}\left(\bar{\epsilon}_2\Gamma_\lambda{}^{\tau\omega}\Gamma^{ijk}\epsilon_1\right)C^{\mu\lambda\rho}C_{\rho\tau\omega}[X^i, X^j, X^k] \ .\end{aligned} \tag{5.1.7}$$

We see that imposing the constraint

$$D_\nu Y^\mu - \frac{\alpha}{6}C^{\mu\lambda\rho}H_{\nu\lambda\rho} = 0 \ , \tag{5.1.8}$$

turns the first term of the closure into a translation. Similarly, with the help of the constraint

$$C^{\mu\nu\sigma}D_\sigma X^i + \frac{3}{\alpha}[Y^\mu, Y^\nu, X^i] = 0 \ , \tag{5.1.9}$$

the second term of the closure represents the first part of a gauge transformation. We see that both these constraints are generalizations of ones found in ([6]).

In order for the third line to turn into the part of a gauge transformation parametrized by $C^{\sigma\tau\omega}$ we need

$$C^{\mu\lambda\rho}\left(\bar{\epsilon}_2\Gamma_{\lambda\rho\sigma}\Gamma^{ij}\epsilon_1\right)Y^\sigma = \frac{6\beta}{\alpha}C^{\sigma\tau\omega}\left(\bar{\epsilon}_2\Gamma_{\sigma\tau\omega}\Gamma^{ij}\epsilon_1\right)Y^\mu \ . \tag{5.1.10}$$

It is easily checked that if $\alpha = 18\beta$ this condition is simply reduced to

$$C \wedge Y = 0 \ . \tag{5.1.11}$$

We will find that the condition $\alpha = 18\beta$ also arises for closure on the other fields. Note that in the condition (5.1.11), Y^μ lives in a 3-algebra, while C does not. A generic choice of C and Y^μ is not immediately compatible with supersymmetry, and condition (5.1.11) is here interpreted as the requirement that Y live in the space orthogonal to C with respect to the wedge product. Work is in progress in order to determine which solutions are consistent with supersymmetry and to understand the form of the (2,0) algebra in such cases.

We require the fourth term to vanish as it parametrizes neither a translation nor a gauge transformation and hence

$$C_{[\mu\nu}{}^\tau C_{\lambda]\tau}{}^\rho = 0 \ . \tag{5.1.12}$$

Note that this means that the components of $C_{\mu\nu\lambda}$ can be identified with the structure constants of a Lie-algebra. Since $\mu, \nu, ... = 0, 1, 2, ...5$ this leads to only two possible choices: $\mathfrak{su}(2)$ and $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

5.1.3 Closure on A_μ

From closing supersymmetry on the gauge field A_μ we expect to find

$$[\delta_1, \delta_2]A_\mu = -v^\nu F_{\mu\nu} + D_\mu\Lambda \ , \tag{5.1.13}$$

Using the relations and constraints found so far, we find after some calculations that

$$\begin{aligned}[\delta_1, \delta_2]\, A_\mu = &- v^\nu \left([Y^\lambda, H_{\mu\nu\lambda},\ \cdot\ ] + \delta(\star C)_{\mu\nu\lambda}[X^i, D^\lambda X^i,\ \cdot\ ] + \frac{i\delta}{2}(\star C)_{\mu\nu\lambda}[\bar{\Psi}, \Gamma^\lambda\Psi,\ \cdot\ ] \right) \\ &+ D_\mu\Lambda + 2i\left(\bar{\epsilon}_2\Gamma_\mu\Gamma^i\epsilon_1\right)\left([Y^\nu, D_\nu X^i,\ \cdot\ ] - (\delta/6)C^{\sigma\tau\omega}[H_{\sigma\tau\omega}, X^i,\ \cdot\ ]\right) \\ &+ 2i(\beta + \delta/6)\left(\bar{\epsilon}_2\Gamma_{[\mu}{}^{\tau\omega}\Gamma^{ijk}\epsilon_1\right) C_{\nu]\tau\omega}[Y^\nu, [X^i, X^j, X^k],\ \cdot\ ] \\ &- i(\bar{\epsilon}_2\Gamma_{\mu\nu\sigma}\Gamma^{ij}\epsilon_1)\left([Y^\nu, [Y^\sigma, X^i, X^j],\ \cdot\ ] + \frac{3\delta}{\alpha}[Y^\nu, [Y^\sigma, X^i, X^j],\ \cdot\ ]\right) .\end{aligned} \tag{5.1.14}$$

We see that in order for the first term to represent a translation we must require the identification

$$F_{\mu\nu}(\cdot) = [Y^\lambda, H_{\mu\nu\lambda},\ \cdot\ ] + \delta(\star C)_{\mu\nu\lambda}[X^i, D^\lambda X^i,\ \cdot\ ] + \frac{i\delta}{2}(\star C)_{\mu\nu\lambda}[\bar{\Psi}, \Gamma^\lambda\Psi,\ \cdot\ ] \ , \tag{5.1.15}$$

which generalizes the constraint in (4.6.3). By looking at the form the closure needs to take, we require the last three terms to vanish. This imposes the correction to the known constraint

$$[Y^\nu, D_\nu X^i,\ \cdot\ ] - \frac{\delta}{6}C^{\sigma\tau\omega}[H_{\sigma\tau\omega}, X^i,\ \cdot\ ] = 0 \ , \tag{5.1.16}$$

as well as the relations between the coefficients

$$\delta = -6\beta, \qquad \alpha = -3\delta \ . \tag{5.1.17}$$

5.1.4 Closure on $H_{\mu\nu\lambda}$

Closing the algebra on $H_{\mu\nu\lambda}$ is somewhat more lengthy, and in the process we found the Mathematica GAMMA package quite helpful ([150]). Supersymmetry should close up to a translation and a gauge transformation

$$[\delta_1, \delta_2]H_{\mu\nu\lambda} = v^\rho D_\rho H_{\mu\nu\lambda} + \Lambda(H_{\mu\nu\lambda}) \ . \tag{5.1.18}$$

Since the calculation is quite involved we will not provide the full details here. Rather we note that in order to close the algebra numerous terms are required to vanish as they parametrize neither a translation, nor a gauge transformation. This is the case if the following relations among the coefficients hold

$$\gamma' = 3\gamma, \qquad \gamma' = 9\beta, \qquad \delta = -2\gamma \ . \tag{5.1.19}$$

Then the remaining terms, making use of the constraints found so far, take the form

$$\begin{aligned}
[\delta_1, \delta_2]\, H_{\mu\nu\lambda} =& v^\rho D_\rho H_{\mu\nu\lambda} - 2i(\bar\epsilon_2 \Gamma_\sigma \Gamma^i \epsilon_1)[Y^\sigma, X^i, H_{\mu\nu\lambda}] \\
&- i\beta\left(\bar\epsilon_2 \Gamma_{\sigma\tau\omega}\Gamma^{ij}\epsilon_1\right) C^{\sigma\tau\omega}[X^i, X^j, H_{\mu\nu\lambda}] \\
&+ 4v^\rho \Bigg(D_{[\lambda} H_{\mu\nu\rho]} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] - \gamma(\star C)_{[\mu\nu\lambda}[X^i, X^j, [Y_{\rho]}, X^i, X^j]] \\
&+ \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar\Psi, \Gamma^\tau \Psi] - i\gamma(\star C)_{[\mu\nu\lambda}[X^i, \bar\Psi, \Gamma_{\rho]}\Gamma^i \Psi]\Bigg) \ ,
\end{aligned} \tag{5.1.20}$$

We see that the first three terms represent a translation and a gauge transformation. The algebra then closes on shell and we find the equation of motion for $H_{\mu\nu\lambda}$

$$\begin{aligned}
D_{[\lambda} H_{\mu\nu\rho]} =& -\frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] + \gamma(\star C)_{[\mu\nu\lambda}[X^i, X^j, [Y_{\rho]}, X^i, X^j]] \\
&- \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar\Psi, \Gamma^\tau \Psi] + i\gamma(\star C)_{[\mu\nu\lambda}[X^i, \bar\Psi, \Gamma_{\rho]}\Gamma^i \Psi] \ .
\end{aligned} \tag{5.1.21}$$

5.1.5 Closure on Ψ

Closure of supersymmetry on the fermion Ψ should be obtained up to a translation and a gauge transformation

$$[\delta_1, \delta_2]\Psi = v^\rho D_\rho \Psi + \Lambda(\Psi) \ . \tag{5.1.22}$$

An explicit calculation, making use of the Gamma package ([150]) and the constraints found so far, gives

$$\begin{aligned}
[\delta_1, \delta_2]\Psi =& v^\rho D_\rho \Psi + \Lambda(\Psi) \\
&+ \frac{3i}{4}(\bar\epsilon_2 \Gamma_\sigma \epsilon_1)\Gamma^\sigma \left(\Gamma^\rho D_\rho \Psi + \Gamma_\rho \Gamma^i [Y^\rho, X^i, \Psi] + \frac{\gamma}{3!}\Gamma_{\rho\sigma\tau}C^{\rho\sigma\tau}\Gamma^{ij}[X^i, X^j, \Psi]\right) \\
&- \frac{i}{4}(\bar\epsilon_2 \Gamma_\sigma \Gamma^j \epsilon_1)\Gamma^\sigma \Gamma^j \left(\Gamma^\rho D_\rho \Psi + \Gamma_\rho \Gamma^i [Y^\rho, X^i, \Psi] + \frac{\gamma}{3!}\Gamma_{\rho\sigma\tau}C^{\rho\sigma\tau}\Gamma^{ij}[X^i, X^j, \Psi]\right) \ .
\end{aligned} \tag{5.1.23}$$

We see that in order to close the algebra the terms other than the translation and the gauge transformation need to vanish. This is achieved upon imposing the Fermion equation of motion, which agrees with (5.1.3).

5.1.6 Bosonic Equations of Motion

We can vary the Fermion equation of motion (5.1.3) to find the equations of motion for X^i and $H_{\mu\nu\lambda}$. We find, making use of the constraints found so far, the following variation

$$\begin{aligned}
&\Big(D^2X^i - \frac{i}{2}[Y^\sigma, \bar{\Psi}, \Gamma_\sigma\Gamma^i\Psi] + [Y^\sigma, X^j, [Y_\sigma, X^j, X^i]] \\
&+ \frac{i\gamma}{3!}C^{\sigma\tau\omega}[\bar{\Psi}, \Gamma_{\sigma\tau\omega}\Gamma^{ij}\Psi, X^j] + \beta\gamma C^{\sigma\tau\omega}C_{\sigma\tau\omega}[[X^i, X^j, X^k], X^j, X^k]\Big)\Gamma^i\epsilon \\
&+ \frac{1}{3!}\Big(D_\mu H_{\nu\lambda\rho} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] - \gamma(\star C)_{\mu\nu\lambda}[X^i, X^j, [Y_\rho, X^i, X^j]] \\
&+ \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar{\Psi}, \Gamma^\tau\Psi] - i\gamma(\star C)_{\mu\nu\lambda}[X^i, \bar{\Psi}, \Gamma_\rho\Gamma^i\Psi]\Big)\Gamma^{\mu\nu\lambda\rho}\epsilon = 0\ . \qquad (5.1.24)
\end{aligned}$$

We see that the equation of motion for $H_{\mu\nu\lambda}$ agrees with the one found by requiring closure of the algebra (5.1.21). Moreover, we find the equation of motion for X^i

$$\begin{aligned}
D^2X^i &= \frac{i}{2}[Y^\sigma, \bar{\Psi}, \Gamma_\sigma\Gamma^i\Psi] - [Y^\sigma, X^j, [Y_\sigma, X^j, X^i]] \\
&- \frac{i\gamma}{3!}C^{\sigma\tau\omega}[\bar{\Psi}, \Gamma_{\sigma\tau\omega}\Gamma^{ij}\Psi, X^j] - \beta\gamma C^{\sigma\tau\omega}C_{\sigma\tau\omega}[[X^i, X^j, X^k], X^j, X^k]\ . \qquad (5.1.25)
\end{aligned}$$

Therefore we have determined the equations of motion for all the degrees of freedom of the (2,0) tensor multiplet.

5.1.7 Summary

We have shown that the (2,0) algebra (5.1.1) we proposed closes on shell. We found corrections to the equations of motion and constraints (4.6.3), which we list here for convenience. Since we are free to rescale $C_{\mu\nu\lambda}$ we can, without loss of generality, set the coefficients of the (2,0) algebra to the specific values

$$\alpha = 3 \quad \beta = 1/3! \quad \gamma = 1/2 \quad \delta = -1 \quad \gamma' = 3/2\ , \qquad (5.1.26)$$

which respect the relations found in the closure of the algebra. The equations of motion for the fields of the tensor multiplet are

$$\begin{aligned}
0 &= D^2X^i - \frac{i}{2}[Y^\sigma, \bar{\Psi}, \Gamma_\sigma\Gamma^i\Psi] + [Y^\sigma, X^j, [Y_\sigma, X^j, X^i]] \\
&+ \frac{i}{2\cdot 3!}C^{\sigma\tau\omega}[\bar{\Psi}, \Gamma_{\sigma\tau\omega}\Gamma^{ij}\Psi, X^j] + \frac{1}{2\cdot 3!}C^{\sigma\tau\omega}C_{\sigma\tau\omega}[[X^i, X^j, X^k], X^j, X^k] \\
0 &= D_{[\lambda}H_{\mu\nu\rho]} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, X^i, D^\tau X^i] - \frac{1}{2}(\star C)_{[\mu\nu\lambda}[X^i, X^j, [Y_{\rho]}, X^i, X^j]] \\
&+ \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[Y^\sigma, \bar{\Psi}, \Gamma^\tau\Psi] - \frac{i}{2}(\star C)_{[\mu\nu\lambda}[X^i, \bar{\Psi}, \Gamma_{\rho]}\Gamma^i\Psi] \\
0 &= \Gamma^\rho D_\rho\Psi + \Gamma_\rho\Gamma^i[Y^\rho, X^i, \Psi] + \frac{1}{2\cdot 3!}\Gamma_{\rho\sigma\tau}C^{\rho\sigma\tau}\Gamma^{ij}[X^i, X^j, \Psi]\ , \qquad (5.1.27)
\end{aligned}$$

while the additional constraints for the algebra to close on shell are

$$
\begin{aligned}
0 &= F_{\mu\nu}(\cdot) - [Y^\lambda, H_{\mu\nu\lambda}, \ \cdot\ ] + (\star C)_{\mu\nu\lambda}[X^i, D^\lambda X^i, \ \cdot\ ] + \frac{i}{2}(\star C)_{\mu\nu\lambda}[\bar\Psi, \Gamma^\lambda \Psi, \ \cdot\ ] \\
0 &= D_\nu Y^\mu - \frac{1}{2} C^{\mu\lambda\rho} H_{\nu\lambda\rho} \\
0 &= C^{\mu\nu\sigma} D_\sigma(\cdot) + [Y^\mu, Y^\nu, \ \cdot\ ] \\
0 &= [Y^\nu, D_\nu \cdot, \cdot'] + \frac{1}{3!} C^{\sigma\tau\omega}[H_{\sigma\tau\omega}, \ \cdot\ , \ \cdot'\ ] \\
0 &= C \wedge Y \ .
\end{aligned}
\tag{5.1.28}
$$

The equations of motion (5.1.27) are invariant under the (2,0) supersymmetry realised by the variations

$$
\begin{aligned}
\delta X^i &= i\bar\epsilon \Gamma^i \Psi \\
\delta Y^\mu &= \frac{i}{2}\bar\epsilon \Gamma_{\lambda\rho} C^{\mu\lambda\rho}\Psi \\
\delta \Psi &= \Gamma^\mu \Gamma^i D_\mu X^i \epsilon + \frac{1}{2\cdot 3!} H_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon \\
&\quad - \frac{1}{2}\Gamma_\mu \Gamma^{ij}[Y^\mu, X^i, X^j]\epsilon + \frac{1}{3!^2} C_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\Gamma^{ijk}[X^i, X^j, X^k]\epsilon \\
\delta H_{\mu\nu\lambda} &= 3i\bar\epsilon \Gamma_{[\mu\nu} D_{\lambda]}\Psi + i\bar\epsilon \Gamma^i \Gamma_{\mu\nu\lambda\rho}[Y^\rho, X^i, \Psi] \\
&\quad + \frac{i}{2}\bar\epsilon(\star C)_{\mu\nu\lambda}\Gamma^{ij}[X^i, X^j, \Psi] + \frac{3i}{4}\bar\epsilon \Gamma_{[\mu\nu|\rho\sigma} C^{\rho\sigma}{}_{\lambda]}\Gamma^{ij}[X^i, X^j, \Psi] \\
\delta A_\mu(\cdot) &= i\bar\epsilon \Gamma_{\mu\nu}[Y^\nu, \Psi, \ \cdot\ ] - \frac{i}{3!}\bar\epsilon C^{\nu\lambda\rho}\Gamma_{\mu\nu\lambda\rho}\Gamma^i[X^i, \Psi, \ \cdot\ ] \ .
\end{aligned}
\tag{5.1.29}
$$

5.2 Conserved Currents

In this section we construct the supercurrent S^μ and energy-momentum tensor $T_{\mu\nu}$ associated to the supersymmetry algebra realised in (5.1.1). We can then deduce the form of the superalgebra including the central charges.

The supercurrent can be easily computed by

$$
\bar\epsilon S^\mu = 2\pi i \langle \overline{\delta_\epsilon \Psi}, \Gamma^\mu \Psi \rangle \ . \tag{5.2.1}
$$

Note the pre-factor of 2π which is needed to produce the correct energy-momentum tensor and will be justified in due course. Explicitly we find

$$
\begin{aligned}
S^\mu =& -2\pi i \langle D_\nu X^i, \Gamma^\nu \Gamma^i \Gamma^\mu \Psi\rangle + \frac{\pi i}{3!}\langle H_{\sigma\tau\omega}, \Gamma^{\sigma\tau\omega}\Gamma^\mu \Psi\rangle - \pi i \langle [Y_\nu, X^i, X^j], \Gamma^\nu \Gamma^{ij}\Gamma^\mu \Psi\rangle \\
&+ \frac{\pi i}{3\cdot 3!} C_{\sigma\tau\omega}\langle [X^i, X^j, X^k], \Gamma^{ijk}\Gamma^{\sigma\tau\omega}\Gamma^\mu \Psi\rangle \ .
\end{aligned}
\tag{5.2.2}
$$

The supercurrent is indeed found to be conserved on shell.

Next we construct the energy-momentum tensor, which after some trial and error, reads

$$\begin{aligned}T_{\mu\nu} =&2\pi\langle D_\mu X^i, D_\nu X^i\rangle - \pi\eta_{\mu\nu}\langle D_\lambda X^i, D^\lambda X^i\rangle + \pi\langle[X^i,X^j,Y_\mu],[X^i,X^j,Y_\nu]\rangle\\ &-\frac{\pi}{2}\eta_{\mu\nu}\langle[X^i,X^j,Y_\lambda],[X^i,X^j,Y^\lambda]\rangle + \frac{\pi}{2}\langle H_{\mu\lambda\rho}, H_\nu{}^{\lambda\rho}\rangle - i\pi\langle\bar\Psi,\Gamma_\mu D_\nu\Psi\rangle\\ &-i\pi\langle\bar\Psi,\Gamma_\nu D_\mu\Psi\rangle + i\pi\eta_{\mu\nu}\langle\bar\Psi,\Gamma^\lambda D_\lambda\Psi\rangle - i\pi\eta_{\mu\nu}\langle[\bar\Psi,Y^\lambda,X^i],\Gamma_\lambda\Gamma^i\Psi\rangle\\ &+\frac{\pi}{3!}\langle[X^i,X^j,X^k],[X^i,X^j,X^k]\rangle(C_{\mu\tau\omega}C_\nu{}^{\tau\omega} - \frac{1}{3!}\eta_{\mu\nu}C^2)\\ &+\frac{\pi}{3!}C_{\mu\lambda\rho}(\star C)_\nu{}^{\lambda\rho}\langle[X^i,X^j,X^k],[X^i,X^j,X^k]\rangle - \frac{i\pi}{3!}\eta_{\mu\nu}C^{\sigma\tau\omega}\langle[\bar\Psi,\Gamma_{\sigma\tau\omega}\Gamma^{ij}\psi,X^i],X^j\rangle\ .\end{aligned} \tag{5.2.3}$$

The energy-momentum tensor is found to satisfy $\partial^\mu T_{\mu\nu} = 0$ using the equations of motion and constraints for the fields derived in the previous section. Although we note that the bosonic part is not symmetric for a general choice of three-form due to the $C_{\mu\lambda\rho}(\star C)_\nu{}^{\lambda\rho}$ term (as well as the more familiar asymmetry arising from the fermions). The 2π prefactor was justified in ([151]) to agree with charge quantization and also in ([144]) to reproduce the correct energy density for M2-branes ending on M5-branes. It also leads to the correct matching of instanton-solitons with KK tower modes ([144]).

In order to derive the super-algebra we make use of the the chain of identities

$$i\bar\epsilon^B\{Q_A,Q_B\} = i\{\bar\epsilon Q, Q_A\} = \delta_\epsilon Q_A = \int d^5x\ (\delta_\epsilon S^0)_A\ , \tag{5.2.4}$$

where

$$Q = \int d^5x\ S^0\ . \tag{5.2.5}$$

Since by construction $\{Q_A,Q_B\}$ is symmetric in A,B, we can extract the momentum

$$P_\nu = \int d^5x T_{0\nu}\ , \tag{5.2.6}$$

and the central charges $(Z^i_\mu, Z^{ij}_{\mu\nu\lambda})$ following the expansion

$$\{Q_A,Q_B\} = 2(\Gamma^\mu C^{-1})_{AB}P_\mu + (\Gamma^\mu\Gamma^i C^{-1})_{AB}Z^i_\mu + \frac{1}{2!\cdot 3!}(\Gamma^{\mu\nu\lambda}\Gamma^{ij}C^{-1})_{AB}Z^{ij}_{\mu\nu\lambda}\ . \tag{5.2.7}$$

In case of vanishing Fermions, we find the following central charges. For Z^i_μ we find

$$Z^i_0 = 4\pi\int d^5x\ \langle[Y_0,X^i,X^j],D^0X^j\rangle - \langle[Y_{\dot\mu},X^i,X^j],D^{\dot\mu}X^j\rangle \tag{5.2.8}$$

$$\begin{aligned}Z^i_{\dot\mu} =&4\pi\int d^5x\ \langle[Y^0,X^i,X^j],D_{\dot\mu}X^j\rangle + \langle[Y_{\dot\mu},X^i,X^j],D^0X^j\rangle\\ &+\langle H_{0\dot\mu\dot\nu},D^{\dot\nu}X^i\rangle + \frac{1}{3}C^+_{0\dot\mu\dot\nu}\langle[X^j,X^k,X^l],D^{\dot\nu}X^m\rangle\varepsilon^{ijklm}\\ &- C^+_{0\dot\mu\dot\nu}\langle[X^i,X^j,X^k],[Y^{\dot\nu},X^j,X^k]\rangle\ ,\end{aligned} \tag{5.2.9}$$

while $Z^{ij}_{\mu\nu\lambda}$ reads (all the expressions should be taken to be anti-symmeterized in i, j and $\dot\mu, \dot\nu, \dot\lambda$ where dotted indices only run over spatial coordinates $\dot\mu, \dot\nu = 1, 2, \ldots, 5$.)

$$\begin{aligned} Z^{ij}_{0\dot\mu\dot\nu} =& 4\pi \int d^5x\ 2\langle [Y_{\dot\mu}, X^i, X^k], [Y_{\dot\nu}, X^k, X^j]\rangle - \langle [Y_{\dot\nu}, X^k, X^l], D_{\dot\mu} X^m\rangle \varepsilon^{ijklm} \\ &+ \frac{1}{2}\langle H_{0\dot\mu\dot\nu}, [Y^0, X^i, X^j]\rangle - \frac{1}{2}\langle H_{\dot\mu\dot\nu\dot\rho}, [Y^{\dot\rho}, X^i, X^j]\rangle - 2\langle D_{\dot\mu} X^i, D_{\dot\nu} X^j\rangle \\ &- \langle (C_{\dot\mu\dot\nu\dot\rho} D^{\dot\rho} X^k + C_{0\dot\mu\dot\nu} D_0 X^k), [X^i, X^j, X^k]\rangle \\ &+ \frac{1}{2}\langle (C_{\dot\mu\dot\nu\dot\rho}[Y^{\dot\rho}, X^k, X^n] - C_{0\dot\mu\dot\nu}[Y^0, X^k, X^n]), [X^l, X^m, X^n]\rangle \varepsilon^{ijklm} \\ &- \frac{1}{2\cdot 3!}\langle [X^k, X^l, X^m], (2C_{0\dot\nu\dot\rho} H_{0\dot\mu}{}^{\dot\rho} + C_{\dot\nu\dot\rho\dot\sigma} H_{\dot\mu}{}^{\dot\rho\dot\sigma})\rangle \varepsilon^{ijklm} \end{aligned} \tag{5.2.10}$$

$$\begin{aligned} Z^{ij}_{\dot\mu\dot\nu\dot\lambda} =& 4\pi \int d^5x\ \frac{1}{2}\langle H_{\dot\mu\dot\nu\dot\lambda}, [Y^0, X^i, X^j]\rangle - \frac{3}{2}\langle H_{0\dot\mu\dot\nu}, [Y_{\dot\lambda}, X^i, X^j]\rangle \\ &- \langle (C_{\dot\mu\dot\nu\dot\lambda} D_0 X^k + 3C_{0\dot\mu\dot\nu} D_{\dot\lambda} X^k), [X^i, X^j, X^k]\rangle \\ &- \frac{1}{2}\langle (C_{\dot\mu\dot\nu\dot\lambda}[Y_0, X^m, X^n] + 3C_{0\dot\mu\dot\nu}[Y_{\dot\lambda}, X^m, X^n]), [X^k, X^l, X^n]\rangle \varepsilon^{ijklm} \\ &+ \frac{1}{4}\langle (C_{\dot\mu\dot\nu\dot\rho} H_{0\dot\lambda}{}^{\dot\rho} - C_{0\dot\lambda\dot\rho} H_{\dot\mu\dot\nu}{}^{\dot\rho}), [X^k, X^l, X^m]\rangle \varepsilon^{ijklm}\ . \end{aligned} \tag{5.2.11}$$

5.3 From (2,0) to 2 M2's

As recalled in section 2 previous work has examined the dynamical systems that arise from the above system when $C_{\mu\nu\lambda}$ vanishes ([6, 129, 144]). To this end let us split up spacetime into the directions $\alpha, \beta = 0, 1, 2$ and $a, b = 3, 4, 5$ and fix

$$C_{abc} = l^3\,\varepsilon_{abc}\ , \tag{5.3.1}$$

where l has dimension of length. This breaks to the $SO(1,5)$ Lorentz symmetry to $SO(1,2) \times SO(3)$. We will see that this $SO(3)$ enhances the $SO(5)$ R-symmetry to $SO(8)$.

Recall the constraints found upon closing the (2,0) algebra (5.1.1) on the tensor multiplet

$$\begin{aligned} 0 &= F_{\mu\nu}(\cdot) - [Y^\lambda, H_{\mu\nu\lambda},\ \cdot\ ] + (\star C)_{\mu\nu\lambda}[X^i, D^\lambda X^i,\ \cdot\ ] + \frac{i}{2}(\star C)_{\mu\nu\lambda}[\bar\Psi, \Gamma^\lambda\Psi,\ \cdot\ ] \\ 0 &= D_\nu Y^\mu - \frac{1}{2}\, C^{\mu\lambda\rho} H_{\nu\lambda\rho} \\ 0 &= C^{\mu\nu\sigma} D_\sigma(\cdot) + [Y^\mu, Y^\nu,\ \cdot\ ] \\ 0 &= [Y^\nu, D_\nu \cdot, \cdot'] + \frac{1}{3!}\, C^{\sigma\tau\omega}[H_{\sigma\tau\omega},\ \cdot\ ,\ \cdot'\ ]\ . \end{aligned} \tag{5.3.2}$$

We now look at the third constraint

$$C^{\mu\nu\sigma} D_\sigma(\cdot) + [Y^\mu, Y^\nu,\ \cdot\ ] = 0\ , \tag{5.3.3}$$

The simplest way to solve this constraint is to take the fields independent of the the x^a spatial directions: $\partial_a(\cdot) = 0$. Then the constraint is solved for

$$A_a(\cdot) = \frac{1}{2l^3}\varepsilon_{abc}[Y^b, Y^c, \ \cdot\ ] \ . \tag{5.3.4}$$

Next we look at the last constraint

$$[Y^\nu, D_\nu \ \cdot\ , \ \cdot' \ ] + \frac{1}{6}C^{\sigma\tau\omega}[H_{\sigma\tau\omega}, \ \cdot\ , \ \cdot' \ ] = 0 \ , \tag{5.3.5}$$

and we see that a solution is given by

$$Y^\alpha = 0 \qquad H_{abc} = -\frac{1}{l^6}[Y_a, Y_b, Y_c] \ , \tag{5.3.6}$$

where to obtain the last relation we used the fundamental identity. Note that the second constraint is also solved by (5.3.6). Finally the first constraint is satisfied if in addition we have

$$H_{\alpha ab} = \frac{1}{l^3}\varepsilon_{abc}D_\alpha Y^c \ . \tag{5.3.7}$$

We note that similar expressions for $H_{\mu\nu\lambda}$ appeared in ([147]). We also find that

$$F_{\alpha\beta}(\cdot) = -\frac{1}{l^3}\varepsilon_{\alpha\beta\gamma}[Y_a, D^\gamma Y^a, \ \cdot\ ] - l^3\varepsilon_{\alpha\beta\gamma}[X^i, D^\gamma X^i, \ \cdot\ ] - \frac{il^3}{2}\varepsilon_{\alpha\beta\gamma}[\bar\Psi, \Gamma^\gamma\Psi, \ \cdot\ ] \ . \tag{5.3.8}$$

To summarise, we found a solution to the constraints (5.3.2) given by

$$\begin{aligned}
\partial_a(\cdot) &= Y^\alpha = 0 \\
A_a(\cdot) &= \frac{1}{2l^3}\varepsilon_{abc}[Y^b, Y^c, \ \cdot\ ] \\
F_{\alpha\beta}(\cdot) &= -\frac{1}{l^3}\varepsilon_{\alpha\beta\gamma}[Y_a, D^\gamma Y^a, \ \cdot\ ] - l^3\varepsilon_{\alpha\beta\gamma}[X^i, D^\gamma X^i, \ \cdot\ ] - \frac{il^3}{2}\varepsilon_{\alpha\beta\gamma}[\bar\Psi, \Gamma^\gamma\Psi, \ \cdot\ ] \\
H_{abc} &= -\frac{1}{l^6}[Y_a, Y_b, Y_c] \\
H_{\alpha ab} &= \frac{1}{l^3}\varepsilon_{abc}D_\alpha Y^c \ ,
\end{aligned} \tag{5.3.9}$$

with the other components of $H_{\mu\nu\lambda}$ fixed by self-duality. We now wish to implement the solution to the constraints that we found into the algebra (5.1.29). We see that since the fields are required to be independent of the three spatial directions, a dimensional reduction naturally arises.

Let us now look at the supersymmetry transformations and apply the solution to the constraints (5.3.9). We find, noting that the fields now depend only on x^α, for the fermions

$$\begin{aligned}
\delta\Psi = {}& \Gamma^\alpha\Gamma^i D_\alpha X^i\epsilon + \frac{1}{2l^3}\Gamma^{ab}\Gamma_{345}\Gamma^i[Y^a, Y^b, X^i]\epsilon - \frac{1}{3!l^6}\Gamma_{abc}[Y^a, Y^b, Y^c]\epsilon \\
& + \frac{1}{l^3}\Gamma^\alpha\Gamma^c\Gamma_{345}D_\alpha Y^c\epsilon - \frac{1}{2}\Gamma^a\Gamma^{ij}[Y^a, X^i, X^j]\epsilon + \frac{1}{3!l^3}\Gamma_{345}\Gamma^{ijk}[X^i, X^j, X^k]\epsilon \ ,
\end{aligned} \tag{5.3.10}$$

and for the bosons

$$\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\
\delta Y^a &= il^3\bar{\epsilon}\Gamma^a\Gamma_{345}\Psi \\
\delta A_\alpha(\cdot) &= i\bar{\epsilon}\Gamma_\alpha\Gamma^b[Y^b,\Psi,\ \cdot\ ] - il^3\bar{\epsilon}\Gamma_\alpha\Gamma_{345}\Gamma^i[X^i,\Psi,\ \cdot\ ]\ .
\end{aligned} \tag{5.3.11}$$

We can now discuss how the degrees of freedom of the two theories are related. The eight scalars parametrizing fluctuations in the directions transverse to the M2-branes worldvolume will consist of the five scalars X^i of the (2,0) tensor multiplet and the three remaining scalars Y^α. Therefore we can define the 3-dimensional scalars:

$$X^I \equiv (l^{-3/2}Y^a, l^{3/2}X^i)\ , \tag{5.3.12}$$

where now $I,J = 3,4,5,...,10$. Note that no other bosonic degrees of freedom are present since $H_{\mu\nu\lambda}$ is fixed by the constraints (5.3.9).

Next we explain how the fermionic degrees of freedom of the two theories are related. Let us define

$$\Omega = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Gamma_{345}\ , \tag{5.3.13}$$

then $\Omega^2 = \Gamma_{345}$ and we see that

$$\Gamma_{012}\Omega = \Omega^{-1}\Gamma_{012}\ . \tag{5.3.14}$$

A consequence of this is that if we define

$$\epsilon' = \Omega\epsilon \qquad \Psi' = l^{3/2}\Omega\Psi\ , \tag{5.3.15}$$

then

$$\Gamma_{012}\epsilon' = \epsilon' \qquad \Gamma_{012}\Psi' = -\Psi'\ , \tag{5.3.16}$$

and hence ϵ' can be thought of as parametrizying the supersymmetries preserved by an M2-brane along x^α.

The supersymmetry transformations now read

$$\begin{aligned}
\delta\Psi' &= \Gamma^\alpha\Gamma^I D_\alpha X^I\epsilon' - \frac{1}{3!}\Gamma^{IJK}[X^I,X^J,X^K]\epsilon' \\
\delta X^I &= i\bar{\epsilon}'\Gamma^I\Psi' \\
\delta A_\alpha(\cdot) &= i\bar{\epsilon}'\Gamma_\alpha\Gamma^I[X^I,\Psi',\ \cdot\ ]\ .
\end{aligned} \tag{5.3.17}$$

These are exactly the variations of the maximally supersymmetric M2-brane model ([4, 142]). Moreover, we see that the constraint (5.3.9) for the field strength $F_{\alpha\beta}$

$$F_{\alpha\beta}(\cdot) = -\varepsilon_{\alpha\beta\gamma}[X^I, D^\gamma X^I,\ \cdot\ ] - \frac{i}{2}\varepsilon_{\alpha\beta\gamma}[\bar{\Psi}',\Gamma^\gamma\Psi',\ \cdot\ ]\ , \tag{5.3.18}$$

is precisely the equation of motion for the field strength of the maximally supersymmetric M2-brane model. Similarly, the remaining equations of motion reduce to the correct equations of motion:

$$
\begin{aligned}
0 =& D^2 X^I + \frac{1}{2}[[X^I, X^J, X^K], X^J, X^K] + \frac{i}{2}[\bar{\Psi}', \Gamma^{IJ}\Psi', X^J] \\
0 =& \Gamma^\alpha D_\alpha \Psi' + \frac{1}{2}\Gamma^{IJ}[\Psi, X^I, X^J] \; .
\end{aligned} \tag{5.3.19}
$$

Therefore we showed that upon imposing the solution of the constraints (5.3.9) on the (2,0) algebra (5.1.29) we obtain the maximally supersymmetric model describing two M2-branes.

Let us briefly mention what happens if we instead take

$$
C_{\alpha\beta\gamma} = l^3 \varepsilon_{\alpha\beta\gamma} \; . \tag{5.3.20}
$$

This is essentially just a double Wick rotation so that the equations are obtained by a suitable Wick rotation. Thus we arrive at a euclidean field theory in 3 dimensions. An inspection of the equations shows that this has an $SO(2,6)$ R-symmetry. We thus obtain a non-abelian 3-dimensional euclidean theory which is suitable to describe an euclidean M2-brane in $(5+6)$-dimensional spacetime, as appears in the work of ([152]).

Chapter 6

M5-branes on S^2

The results of this chapter have their origin in an early collaboration towards the paper ([12]). I would like to thank the authors for allowing part of the results of ([12]) to appear in this thesis, and for acknowledging the collaboration in the paper itself.

Chapter 4 provided some motivation as to why compactifications of the (2,0) theory to lower dimensions might be of interest. Recall that a number of important results were obtained by relating theories living in different dimensions, by considering their origins as compactifications of the (2,0) theory on some manifolds. The AGT correspondence ([134]), was the first such relation to be found. It relates quantities in 4-dimensional, $\mathcal{N} = 2$ theories to quantities in 2d-dimensional Toda theories, which are non-supersymmetric conformal field theories. Indeed, a class of 4-dimensional $\mathcal{N} = 2$ theories can be obtained as compactification of the (2,0) theory on a Riemann surface $\Sigma_{n,g}$ of genus g with possibly puncture defects (so called theories of class $\mathcal{S}$), while Toda theories of type ADE can be obtained by compactifying the (2,0) theory of type ADE on a four-sphere S^4. The partition function of the (2,0) theory on the product manifold $\Sigma_{n,g} \times S^4$ (or the equivalent replacement of S^4 with an Ω background) then allows to identify quantities in the two theories through the factorization of the partition function and the scaling of the volume of either compactification manifolds thanks to conformal invariance.

Following the AGT correspondence, other results followed relating other compactifications of the (2,0) theory on different manifolds. In particular, in Chapter 4 the 3d-3d correspondence of ([135]) and the 4d-2d correspondence of ([138]) were recalled. The latter is the start of a program involving the compactification of the (2,0) theory on arbitrary four-manifolds M_4 (more precisely those which can be seen as coassociative four-cycles in G_2 manifolds). On the other side of the compactification, the authors of ([138]) consider a two-torus T^2 which produces $\mathcal{N} = 4$ Super-Yang-Mills theory on the four-manifold M_4. In this chapter, we extend such analysis and we study the (2,0) theory on a two-sphere S^2. The resulting theory is a sigma model from $\mathbb{R}^4$ into the moduli space of k centered SU(2) monopoles, where k is the number of parallel M5-branes. In order to obtain this result, the two-sphere is seen as a circle fibration of an interval, such that the radius of the fiber vanishes at the two endpoints of the interval. This allows to reduce the (2,0) theory on the circle fiber and successively on the interval. The first step realizes 5-dimensional $\mathcal{N} = 2$ Super-Yang-Mills theory on the space $\mathbb{R}^4 \times I$, where I is the interval. The vanishing of the radius of the fiber at the two endpoints provide specific boundary condditions. The further reduction of 5-dimensional Super-Yang-Mills theory on an interval I does not produce a superconformal field theory, as it is instead the case for theories of class $\mathcal{S}$. In this case the volume of the sphere, or equivalently the length of the interval in our setup, will set the scale of the underlying theory. In particular, the latter will arise by localizing on the fields configurations that preserve supersymmetry along the interval I. These are found to be those configurations for which the three scalars out of the five of the (2,0) theory which transform under the $\mathfrak{so}(3)_R$ resulting from the breaking of the total R-symmetry group, satisfy Nahm equations. Indeed the moduli space of Nahm equations with particular boundary conditions is known to be isomorphic to the moduli space of k centered SU(2) monopoles ([153]). Therefore, by using the intermediate reduction to 5-dimensional Super-Yang-Mills theory allowed by the circle fibration of the interval, it is possible to identify the supersymmetric configurations as those which satisfy Nahm equations on the interval. In this chapter such analysis is carried out, by embedding the (2,0) in an appropriate supergravity background which allows to preserve supersymmetry on the two-sphere. The analysis follows closely the one carried out in ([12]), arising from an early collaboration. In ([12]) the authors go on to study the reduction for a generic four-manifold M_4, while here we only study the case of flat $\mathbb{R}^4$.

The structure of this chapter is the following. In section 6.1 we study the supergravity background preserving supersymmetry on the curved two-sphere. In section 6.2 we carry out the circle reduction from 6 dimensions to Super-Yang-Mills theory in 5 dimensions, and in section 6.3 we reduce to 4 dimensions by shrinking the size of the interval to zero.

6.1 Supergravity Backgrounds and Twists

This section serves two purposes: firstly to explain the possible twists of the 6-dimensional $\mathcal{N} = (0, 2)$ theory on a two-sphere S^2, and secondly, to determine the supergravity background associated to the topological half-twist on S^2.

6.1.1 Twisting on S^2

The R-symmetry and Lorentz algebra of the M5-brane theory are

$$\mathfrak{sp}(4)_R \oplus \mathfrak{so}(6)_L \,. \tag{6.1.1}$$

The supercharges transform in the $(\mathbf{4}, \bar{\mathbf{4}})$ spinor representation (the same representation as the fermions in the theory). The product structure of the space-times implies that we decompose the Lorentz algebra as

$$\mathfrak{so}(6)_L \quad \rightarrow \quad \mathfrak{so}(4)_L \oplus \mathfrak{so}(2)_L \cong \mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r \oplus \mathfrak{so}(2)_L \,. \tag{6.1.2}$$

Consider the decomposition of the R-symmetry as

$$\mathfrak{sp}(4)_R \quad \rightarrow \quad \mathfrak{su}(2)_R \oplus \mathfrak{so}(2)_R \tag{6.1.3}$$

For our analysis we first consider the theory on $S^2 \times \mathbb{R}^4$ and the twist along S^2. The Lorentz and R-symmetry groups reduce again as in (6.1.2) and (6.1.3). The twist is implemented by identifying $\mathfrak{so}(2)_R$ with $\mathfrak{so}(2)_L$ and we denote it $\mathfrak{so}(2)_{twist} \simeq \mathfrak{u}(1)_{twist}$. As we have seen this is compatible with the twist 1, discussed in the last subsection.

$$\text{Twist } S^2: \qquad \mathfrak{so}(6)_L \oplus \mathfrak{sp}(4)_R \quad \rightarrow \quad \mathfrak{g}_{res} \cong \mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_{twist} \,. \tag{6.1.4}$$

The residual symmetry group and decomposition of the supercharges and fermions is then

$$\begin{aligned} \mathfrak{so}(6)_L \oplus \mathfrak{sp}(4)_R \quad &\rightarrow \quad \mathfrak{g}_{res} \cong \mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_{twist} \\ \mathbf{4} \otimes \overline{\mathbf{4}} \quad &\rightarrow \quad (\mathbf{2}, \mathbf{1}, \mathbf{2})_{-2} \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{+\mathbf{2}} \,. \end{aligned} \tag{6.1.5}$$

There are eight supercharges transforming as singlets on S^2 and transforming as Weyl spinors of opposite chirality on M_4 and doublets under the remaining R-symmetry

The fields of the 6-dimensional (2,0) theory decompose as follows

$$\begin{aligned} &\mathfrak{so}(6)_L \oplus \mathfrak{sp}(4)_R \rightarrow \mathfrak{g}_{res} \cong \mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R \\ &\Phi^{\underline{mn}}: \qquad (\mathbf{1}, \mathbf{5}) \rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_{0,2} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_{0,-2} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3})_{0,0} \\ &\rho^{\widehat{m}}_{\underline{m}}: \qquad (\mathbf{4}, \overline{\mathbf{4}}) \rightarrow (\mathbf{1}, \mathbf{2}, \mathbf{2})_{+1,+1} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1,+1} \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2})_{+1,-1} \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2})_{-1,-1} \\ &\mathcal{B}^{\underline{AB}}: \qquad (\mathbf{15}, \mathbf{1}) \rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_{0,0} \oplus (\mathbf{3}, \mathbf{1}, \mathbf{1})_{0,0} \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1})_{0,0} \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1})_{2,0} \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1})_{-2,0} \,. \end{aligned} \tag{6.1.6}$$

After the twist of the $\mathfrak{u}(1)$ symmetries, note that this is not the standard transformation of the 4-dimensional $\mathcal{N} = 2$ hypermultiplet. Twisting with the $\mathfrak{su}(2)_r$ Lorentz with the remaining $\mathfrak{su}(2)_R$, i.e.

$$\mathfrak{su}(2)_{twist} \cong \mathfrak{su}(2)_r \oplus \mathfrak{su}(2)_R \tag{6.1.7}$$

the resulting topological theory has the following matter content

$$
\begin{array}{rccl}
\mathfrak{so}(6)_L \oplus \mathfrak{sp}(4)_R & \to & \mathfrak{g}_{res} \cong \mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_{twist} \oplus \mathfrak{u}(1)_{twist} \\
\Phi^{\underline{mn}}: \quad (\mathbf{1},\mathbf{5}) & \to & (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{3})_0 \\
\rho_{\underline{m}}^{\widehat{m}}: \quad (\mathbf{4},\overline{\mathbf{4}}) & \to & (\mathbf{1},\mathbf{1}\oplus\mathbf{3})_{+2} \oplus (\mathbf{1},\mathbf{1}\oplus\mathbf{3})_0 \oplus (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{2},\mathbf{2})_{-2} \\
\mathcal{B}^{\underline{AB}}: \quad (\mathbf{15},\mathbf{1}) & \to & (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{2},\mathbf{2})_2 \oplus (\mathbf{2},\mathbf{2})_{-2}\,.
\end{array}
\tag{6.1.8}
$$

6.1.2 Supergravity Background Fields

Before describing the details of the reduction, we should summarize our strategy. Our goal is to determine the dimensional reduction of the 6-dimensional $(2,0)$ theory with non-abelian A_n gauge algebra. For the abelian theory, the dimensional reduction is possible, using the equations of motions in 6 dimensions ([154, 155]). However, for the non-abelian case, due to absence of a 6-dimensional formulation of the theory, we have to follow an alternative strategy. Our strategy is much alike to the derivation of complex Chern-Simons theory as the dimensional reduction on an S^3 in ([156]). First note, that the 6-dimensional theory on S^1 gives rise to 5-dimensional $\mathcal{N}=2$ Super-Yang-Mills theory. More generally, the dimensional reduction of the 6-dimensional theory on a circle-fibration gives rise to a 5-dimensiona Super-Yang-Mills theory in a supergravity background ([157]) (for earlier references see ([158, 159])). This theory has a non-abelian extension, consistent with gauge invariance and supersymmetry, which is then conjectured to be the dimensional reduction of the non-abelian 6-dimensional theory.

More precisely, this approach requires first to determine the background of the 6-dimensional abelian theory as described in terms of the $\mathcal{N}=(2,0)$ conformal supergravity theory ([160, 161]). The 5-dimensional background is determined by reduction on the circle fiber, and then non-abelianized. We can then further reduce the theory along the remaining compact directions to determine the theory in 4-dimensional. For S^3, there is the Hopf-fibration, used in ([156]) to derive the Chern-Simons theory in this two-step reduction process. Here, for the S^2, we will fiber the S^1 over an interval, and necessarily, the fibers will have to become singular at the end-points.

In the following we will prepare the analysis of the supergravity background. By requiring invariance under the residual group of symmetries $\mathfrak{g}_{res}$ preserved by the topological twist on S^2, we derive ansätze for the background fields in 6-dimensional $\mathcal{N}=(0,2)$ off-shell conformal supergravity fields. In the next section we will consider the Killing spinor equations and fix the background fields completely.

Label	Field	$\mathfrak{sp}(4)_R$	Properties
$e_{\underline{\mu}}^{\underline{A}}$	Frame	$\mathbf{1}$	
$V_{\underline{A}}^{\hat{B}\hat{C}}$	R-symmetry Gauge Field	$\mathbf{10}$	$V_{\underline{A}}^{\hat{B}\hat{C}} = -V_{\underline{A}}^{\hat{C}\hat{B}}$
$T_{[\underline{BCD}]}^{\hat{A}}$	Auxiliary Three-form	$\mathbf{5}$	$T^{\hat{A}} = - * T^{\hat{A}}$
$D_{(\hat{A}\hat{B})}$	Auxiliary Scalar	$\mathbf{14}$	$D_{\hat{A}\hat{B}} = D_{\hat{B}\hat{A}}$, $D_{\hat{A}}^{\hat{A}} = 0$

Table 6.1: Bosonic background fields for the 6-dimensional (2,0) conformal supergravity.

To begin with, the 6-dimensional metric on $S^2 \times \mathbb{R}^4$ is given by

$$ds^2 \;=\; ds^2_{\mathbb{R}^4} + r^2 d\theta^2 + \ell(\theta)^2 \, d\phi^2 \,, \tag{6.1.9}$$

with $\ell(\theta) = r\sin(\theta)$ for the round two-sphere. More generally, $\ell(\theta)$ can be a function, which is smooth an interpolates between

$$\frac{\ell(\theta)}{r} \sim \theta \,, \quad \text{for} \quad \theta \to 0 \,, \qquad \frac{\ell(\theta)}{r} \sim \pi - \theta \,, \quad \text{for} \quad \theta \to \pi \,. \tag{6.1.10}$$

We choose the frame

$$e^A = dx^A \,, \qquad e^5 = r\, d\theta \,, \qquad e^6 = \ell(\theta)\, d\phi \,. \tag{6.1.11}$$

The corresponding non-vanishing components of the spin connection are

$$\omega^{56} = -\omega^{65} = -\frac{\ell'(\theta)}{r}\, d\phi \,. \tag{6.1.12}$$

In the following the index conventions are such that all hatted indices are R-symmetry, all unhatted Lorentz indices. All our conventions are summarized in appendix B.1. The background fields for the off-shell gravity multiplet are summarized in table 6.1.

Before setting up the ansätze note the following decompositions of representations, that the background fields transform under, first for the Lorentz symmetry,

$$\begin{array}{rrcl} & \mathfrak{so}(6)_L & \to & \mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r \oplus \mathfrak{u}(1)_L \\ \underline{A}: & \mathbf{6} & \to & (\mathbf{2},\mathbf{2})_\mathbf{0} \oplus (\mathbf{1},\mathbf{1})_\mathbf{2} \oplus (\mathbf{1},\mathbf{1})_{-\mathbf{2}} \\ [\underline{BCD}]^{(+)}: & \mathbf{10} & \to & (\mathbf{2},\mathbf{2})_\mathbf{0} \oplus (\mathbf{3},\mathbf{1})_\mathbf{2} \oplus (\mathbf{1},\mathbf{3})_{-\mathbf{2}} \\ [\underline{BC}]: & \mathbf{15} & \to & (\mathbf{2},\mathbf{2})_\mathbf{2} \oplus (\mathbf{2},\mathbf{2})_{-\mathbf{2}} \oplus (\mathbf{3},\mathbf{1})_\mathbf{0} \oplus (\mathbf{1},\mathbf{3})_\mathbf{0} \oplus (\mathbf{1},\mathbf{1})_\mathbf{0} \end{array} \tag{6.1.13}$$

and for the R-symmetry

$$\begin{array}{rrcl} & \mathfrak{so}(5)_R & \rightarrow & \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \\ \widehat{A}: & \mathbf{5} & \rightarrow & \mathbf{3_0} \oplus \mathbf{1_2} \oplus \mathbf{1_{-2}} \\ [\widehat{B}\widehat{C}]: & \mathbf{10} & \rightarrow & \mathbf{3_0} \oplus \mathbf{3_2} \oplus \mathbf{3_{-2}} \oplus \mathbf{1_0} \\ (\widehat{B}\widehat{C}): & \mathbf{14} & \rightarrow & \mathbf{5_0} \oplus \mathbf{3_2} \oplus \mathbf{3_{-2}} \oplus \mathbf{1_2} \oplus \mathbf{1_{-2}} \oplus \mathbf{1_0}\,. \end{array} \tag{6.1.14}$$

The bosonic supergravity fields of 6-dimensional off-shell conformal maximal supergravity ([157, 158, 160–162]) are summarized in appendix are the frame $e_\mu^{\underline{A}}$ and

$$T_{[\underline{BCD}]\widehat{A}}\,, \qquad V_{\underline{A}\,[\widehat{B}\widehat{C}]} \rightarrow (dV)_{[\underline{AB}]\,[\widehat{C}\widehat{D}]}\,, \qquad D_{(\widehat{A}\widehat{B})}\,, \qquad b_{\underline{A}} \rightarrow (db)_{[\underline{AB}]}\,, \tag{6.1.15}$$

where dV and db denote the field strength of the R-symmetry and dilatation gauge fields respectively. Furthermore $T_{\underline{BCD}\widehat{A}}$ is anti-self-dual [1] and $D_{(\widehat{A}\widehat{B})}$ is traceless

$$T_{\underline{BCD}\widehat{A}} = T_{[\underline{BCD}]^{(+)}\widehat{A}}\,, \qquad \delta^{\widehat{A}\widehat{B}} D_{\widehat{A}\widehat{B}} = 0\,. \tag{6.1.16}$$

We shall now decompose these in turn under the residual symmetry group $\mathfrak{g}_{res}$ and determine the invariant components.

1. $T_{\widehat{A}[\underline{BCD}]}$ The decomposition under $\mathfrak{g}_{res}$ (that is, after performing the twist) is given by:

   $$\begin{aligned}(\mathbf{10},\mathbf{5}) \rightarrow & (\mathbf{2},\mathbf{2},\mathbf{3})_{(\mathbf{2})} \oplus (\mathbf{3},\mathbf{1},\mathbf{3})_{(\mathbf{2})} \oplus (\mathbf{1},\mathbf{3},\mathbf{3})_{(-\mathbf{2})} \oplus (\mathbf{2},\mathbf{2},\mathbf{1})_{(\pm\mathbf{2})} \oplus (\mathbf{3},\mathbf{1},\mathbf{1})_{(\mathbf{4})} \\ & \oplus (\mathbf{3},\mathbf{1},\mathbf{1})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{(-\mathbf{4})}\,.\end{aligned} \tag{6.1.17}$$

   This tensor product does not contain any singlet under $\mathfrak{g}_{res}$, so the backgrounds we consider have $T_{\hat{A}\underline{BCD}} = 0$.

2. $V_{\underline{A}[\hat{B}\hat{C}]}$ We are looking for components of the field strength $(dV)_{[\underline{AB}]\,[\hat{C}\hat{D}]}$ invariant under $\mathfrak{g}_{res}$. The decomposition of $(dV)_{[\underline{AB}]\,[\hat{C}\hat{D}]}$ is:

   $$\begin{aligned}(\mathbf{15},\mathbf{10}) \rightarrow & (\mathbf{2},\mathbf{2},\mathbf{3})_{(\pm\mathbf{2})} \oplus (\mathbf{3},\mathbf{1},\mathbf{3})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{3},\mathbf{3})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_{(\mathbf{0})} \oplus (\mathbf{2},\mathbf{2},\mathbf{3})_{(\pm\mathbf{4})} \\ & \oplus 2\times(\mathbf{2},\mathbf{2},\mathbf{3})_{(\mathbf{0})} \oplus (\mathbf{3},\mathbf{1},\mathbf{3})_{(\pm\mathbf{2})} \oplus (\mathbf{1},\mathbf{3},\mathbf{3})_{(\pm\mathbf{2})} \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_{(\pm\mathbf{2})} \\ & \oplus (\mathbf{2},\mathbf{2},\mathbf{1})_{(\pm\mathbf{2})} \oplus (\mathbf{3},\mathbf{1},\mathbf{1})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{(\mathbf{0})}\,.\end{aligned} \tag{6.1.18}$$

[1] In Euclidean signature, $T_{[\underline{BCD}]\widehat{A}}$ can be complexified and taken to satify $T = i * T$.

We see that we have a singlet that corresponds to turning on a flux on the S^2 and an ansatz for V is given by

$$V_{\phi\,\widehat{x}\widehat{y}} = \frac{1}{2}\,v(\theta)\,\epsilon_{\widehat{x}\widehat{y}}\,. \tag{6.1.19}$$

where $\widehat{x}, \widehat{y}$ run over the components $\widehat{B}, \widehat{C} = 4, 5$, and the other components of V vanish.

3. $b_{\underline{A}}$ The field strength $(db)_{[\underline{AB}]}$ decomposes under $\mathfrak{g}_{res}$ as

$$(\mathbf{15},\mathbf{1}) \to (\mathbf{2},\mathbf{2},\mathbf{1})_{(\pm\mathbf{2})} \oplus (\mathbf{3},\mathbf{1},\mathbf{1})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{(\mathbf{0})}\,. \tag{6.1.20}$$

There is a singlet, which corresponds to turning on a field strength on the S^2. In the following we will not consider this possibility. Note that any other choice can always be obtained by a conformal transformation with K, which shifts $b_{\underline{A}}$ ([161]). In the following we thus set

$$b_{\underline{A}} = 0\,. \tag{6.1.21}$$

4. $D_{(\widehat{A}\widehat{B})}$ The decomposition under $\mathfrak{g}_{res}$ is given by

$$(\mathbf{1},\mathbf{14}) \to (\mathbf{1},\mathbf{1},\mathbf{5})_{(\mathbf{0})} \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_{(\pm\mathbf{2})} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{(\pm\mathbf{2})} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{(\mathbf{0})}\,. \tag{6.1.22}$$

There is one singlet corresponding to the ansatz:

$$D_{\widehat{a}\widehat{b}} = d\,\delta_{\widehat{a}\widehat{b}} \quad , \quad D_{\widehat{x}\widehat{y}} = -\frac{3}{2}\,d\,\delta_{\widehat{x}\widehat{y}}\,, \tag{6.1.23}$$

with other components vanishing. The relative coefficients are fixed by the tracelessness condition on $D_{\widehat{A}\widehat{B}}$.

6.1.3 Killing spinors

With the ansätze for the supergravity background fields we can now determine the conditions on the coefficients v and d, to preserve supersymmetry. The background on 6-dimensional supergravity is summarized in section 6.1.2 and the Killing spinor equations (B.2.1) and (B.2.7) are solved in appendix B.2. In summary the background with $t = b = 0$ preserves half the supersymmetries if

$$\begin{aligned} v(\theta) &= -\frac{\ell'(\theta)}{r} \\ d(\theta) &= \frac{3}{2}\frac{\ell''(\theta)}{r^2\ell(\theta)}\,, \end{aligned} \tag{6.1.24}$$

where for the round two-sphere $l(\theta) = r\sin(\theta)$, and the Killing spinor ε is constant and satisfies the following constraint

$$(\Gamma^{\widehat{45}})^{\widehat{m}}{}_{\widehat{n}}\varepsilon^{\widehat{n}} - \Gamma^{56}\varepsilon^{\widehat{m}} = 0\,. \tag{6.1.25}$$

For the round two-sphere $\ell(\theta) = r\sin(\theta)$.

The value of the R-symmetry gauge field $V^{56} = -\frac{\ell'(\theta)}{r}d\phi = \omega^{56}$ and the fact that the preserved supersymmetries are generated by constant spinors indicates that this supergravity background realizes the topological twist on S^2, as expected.

Finally, recall that we chose a gauge for which $b_{\underline{\mu}} = 0$. Note that the background field $b_{\underline{\mu}}$ can be fixed to an arbitrary other value by a special conformal transformation (see ([161])). The special conformal transformation does not act on the other background fields (they transform as scalars under these transformations), nor on the spinor $\varepsilon^{\widehat{m}}$, however it changes the spinor $\eta^{\widehat{m}}$ parametrizing conformal supersymmetry transformations. Indeed one can show that the Killing spinor equations (B.2.1), (B.2.7) are solved for an arbitrary b_μ by the same solution $\varepsilon^{\widehat{m}}$ together with

$$\eta^{\widehat{m}} \;=\; -\frac{1}{2}b_{\underline{A}}\Gamma^{\underline{A}}\varepsilon^{\widehat{m}} \;. \tag{6.1.26}$$

In this way one can recover the gauge choice $b_{\underline{\mu}} = \alpha^{-1}\partial_\mu\alpha$ (with $\alpha = 1/\ell$ in our conventions) of ([157]), although we will keep our more convenient choice $b_{\underline{\mu}} = 0$. For our gauge choice, the dimensional reduction to 5 dimensions is rederived in appendix B.3.

6.2 From 6-dimensional $(2,0)$ on S^2 to 5-dimensional Super-Yang-Mills theory

We now proceed with the dimensional reduction of the 6-dimensional $\mathcal{N} = (2,0)$ theory on S^1 to obtain 5-dimensional $\mathcal{N} = 2$ Super-Yang-Mills theory, as in ([157, 158]). Note that the reduction of the (2,0) theory on S^1 is well known, also for circle fibrations ([163]) and in the case in which a supergravity background is turned on ([157]). However, we proceed to a new analysis of the reduction. We find that a different solution to the Killing-Spinor equations compared to the one found in ([157]) is possible, which corresponds to the gauge in which $b_\mu = 0$. This, together with (6.1.26), implies that $\eta^{\widehat{m}} = 0$. We find this gauge to be much more suitable to work with, in particular in the following step of the reduction to 4 dimensions.

We should remark on an important point in the signature conventions: the reduction to the 5-dimensiona Super-Yang-Mills theory is accomplished in Lorentzian signature, $\mathbb{R}^4 \to \mathbb{R}^{1,3}$, where fields admit 6-dimensional reality conditions, however it would go through in Euclidean signature upon complexifying the fields in 6 dimensions and then imposing reality conditions in 5d. This amounts to Wick-rotating the Lorentzian 5-dimensional theory. In later sections, when we study the 5-dimensional theory on a generic M_4, we adopt the Euclidean signature, which is compatible with the twist on M_4.

6.2.1 The 6-dimensional $(2,0)$ Theory

The abelian 6-dimensional $\mathcal{N}=(0,2)$ theory contains a tensor multiplet, which is comprised of a two-form B with field strength $H = dB$, five scalars $\Phi^{\widehat{m}\widehat{n}}$, and four Weyl spinors $\rho^{\widehat{m}}_{\underline{m}}$ of negative chirality, which are symplectic Majorana. the scalars satisfy $\Phi^{\widehat{m}\widehat{n}} = -\Phi^{\widehat{n}\widehat{m}}$ and $\Omega^{\widehat{m}\widehat{n}}\Phi_{\widehat{m}\widehat{n}} = 0$. The equations of motion are[2]

$$\begin{aligned} H^-_{\underline{\mu\nu\sigma}} - \frac{1}{2}\Phi_{\widehat{m}\widehat{n}} T^{\widehat{m}\widehat{n}}_{\underline{\mu\nu\sigma}} &= 0 \\ \mathcal{D}^2\Phi_{\widehat{m}\widehat{n}} - \frac{1}{15} D^{\widehat{r}\widehat{s}}_{\widehat{m}\widehat{n}}\Phi_{\widehat{r}\widehat{s}} + \frac{1}{3} H^+_{\underline{\mu\nu\sigma}} T^{\underline{\mu\nu\sigma}}_{\widehat{m}\widehat{n}} &= 0 \\ \not{\mathcal{D}}\rho^{\widehat{m}} - \frac{1}{12} T^{\widehat{m}\widehat{n}}_{\underline{\mu\nu\sigma}}\Gamma^{\underline{\mu\nu\sigma}}\rho_{\widehat{n}} &= 0. \end{aligned} \tag{6.2.1}$$

Here $H^{\pm} = 1/2(H \pm \star H)$ and the R-symmetry indices of the background fields have been transformed $\widehat{A} \to \widehat{m}\widehat{n}$ using the Gamma matrices as in (B.2.3). The covariant derivatives are defined as follows

$$\begin{aligned} D_{\underline{\mu}}\rho^{\widehat{m}} &= \left(\partial_{\underline{\mu}} - \frac{5}{2}b_{\underline{\mu}} + \frac{1}{4}\omega^{\underline{AB}}_{\underline{\mu}}\Gamma_{\underline{AB}}\right)\rho^{\widehat{m}} - \frac{1}{2}V^{\widehat{m}}_{\underline{\mu}\widehat{n}}\rho^{\widehat{n}} \\ D_{\underline{\mu}}\Phi^{\widehat{m}\widehat{n}} &= (\partial_{\underline{\mu}} - 2b_{\underline{\mu}})\Phi^{\widehat{m}\widehat{n}} + V^{[\widehat{m}}_{\underline{\mu}\widehat{r}}\Phi^{\widehat{n}]\widehat{r}} \\ D^2\Phi^{\widehat{m}\widehat{n}} &= (\partial^{\underline{A}} - 3b^{\underline{A}} + \omega^{\underline{BA}}_{\underline{B}})D_{\underline{A}}\Phi^{\widehat{m}\widehat{n}} + V^{\underline{\mu}[\widehat{m}}_{\widehat{r}} D_{\underline{\mu}}\Phi^{\widehat{n}]\widehat{r}} - \frac{R_{6d}}{5}\Phi^{\widehat{m}\widehat{n}}. \end{aligned} \tag{6.2.2}$$

Here $R_{6d}$is the 6-dimensional Ricci scalar. These equations are invariant under the supersymmetry transformations

$$\begin{aligned} \delta\mathcal{B}_{\underline{\mu\nu}} &= -\bar{\epsilon}\Gamma_{\underline{\mu\nu}}\rho \\ \delta\Phi_{\hat{m}\hat{n}} &= -4\bar{\epsilon}^{[\hat{m}}\rho^{\hat{n}]} - \Omega^{\hat{m}\hat{n}}\bar{\epsilon}\,\rho \\ \delta\rho^{\hat{m}} &= \frac{1}{48}H^+_{\underline{\mu\nu\sigma}}\Gamma^{\underline{\mu\nu\sigma}}\epsilon^{\hat{m}} + \frac{1}{4}\not{\mathcal{D}}\Phi^{\hat{m}\hat{n}}\epsilon_{\hat{n}} - \Phi^{\widehat{m}\widehat{n}}\eta_{\widehat{n}}\,. \end{aligned} \tag{6.2.3}$$

In appendix B.3 we reduce the equations of motion and obtain a 5-dimensional Super-Yang-Mills in a general background, but for our gauge choice $b_{\underline{\mu}} = 0$, which is a different gauge choice from e.g. ([157]). In this section we specify to the background $\mathbb{R}^4 \times S^2$ and couple this theory to the background supergravity fields of Section 6.1. Using the index convetions in appendix B.1 the 6-dimensional fields are decomposed in the following way

$$\begin{aligned} H &\to F = dA \\ \rho^{\underline{m}\widehat{m}} &\to \begin{pmatrix} 0 \\ i\rho^{m'\widehat{m}} \end{pmatrix} \\ \Phi^{\widehat{m}\widehat{n}} &\to \Phi^{\widehat{m}\widehat{n}}\,, \end{aligned} \tag{6.2.4}$$

[2] We will use the conventions of ([161]).

while the frame is decomposed as follows

$$e_{\underline{A}}^{\mu} \rightarrow \begin{pmatrix} e_{A'}^{\mu'} & e_{A'}^{\phi} \equiv C_{A'} \\ e_{6}^{\mu'} \equiv 0 & e_{6}^{\phi} \equiv \alpha \end{pmatrix} \tag{6.2.5}$$

The action of abelian 5-dimensional Super-Yang-Mills theory in a general background is

$$S_{5d} = S_F + S_{\text{scalar}} + S_\rho, \tag{6.2.6}$$

where

$$\begin{aligned} S_F &= -\int \operatorname{tr}[\alpha \tilde{F} \wedge *_{5d} \tilde{F} + C \wedge F \wedge F] \\ \mathrm{S}_{\text{scalar}} &= -\int d^5x \, \sqrt{|g|}\, \alpha^{-1} \left(\mathcal{D}_{A'} \Phi^{\widehat{m}\widehat{n}} \mathcal{D}^{A'} \Phi_{\widehat{m}\widehat{n}} + 4\Phi^{\widehat{m}\widehat{n}} F_{A'B'} T^{A'B'}_{\widehat{m}\widehat{n}} - \Phi_{\widehat{m}\widehat{n}} (M_\Phi)^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}} \Phi^{\widehat{r}\widehat{s}} \right) \\ S_\rho &= -\int d^5x \sqrt{|g|}\, \alpha^{-1} \rho_{m\widehat{m}} \left(i \not{\mathcal{D}}^m_{\ n} \rho^{n\widehat{m}} + (M_\rho)^{m\widehat{m}}_{n\widehat{n}} \rho^{n\widehat{n}} \right) , \end{aligned} \tag{6.2.7}$$

with all the mass matrices defined in appendix B.3 and $\tilde{F}$ defined as

$$\tilde{F} = F - \frac{1}{\alpha} \Phi_{\widehat{m}\widehat{n}} T^{\widehat{m}\widehat{n}}. \tag{6.2.8}$$

6.2.2 5-dimensional Super-Yang-Mills theory in the Supergravity Background

We now determine the 5-dimensional Super-Yang-Mills theory in the background, which corresponds to the 6-dimensional $(2,0)$ theory on S^2, by performing the dimensional reduction along the circle fiber. As shown in section 6.1.2, the only background fields for the 5-dimensional Super-Yang-Mills theory, which are compatible with the symmetry group, were $D^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}}$ and $V^{\widehat{m}\widehat{n}}_{\phi}$. In this section we use the results from appendix B.3 to derive the action with only the background fields $D^{\widehat{m}\widehat{n}}_{rs}$ and $S^{\widehat{m}\widehat{n}}$ switched on, in the gauge $b_{\underline{\mu}} = 0$.

For our background the metric, graviphoton, $C_{A'}$, and the dilaton, α, are given by

$$ds_5^2 = ds_{\mathbb{R}^4}^2 + r^2 d\theta^2 \,, \quad C_{A'} = 0 \,, \quad \alpha = \frac{1}{\ell(\theta)} \,, \quad 0 \leq \theta \leq \pi \,, \tag{6.2.9}$$

which means that $G = dC = 0$. Imposing these conditions and turning on only the background fields $D^{\widehat{m}\widehat{n}}_{rs}$ and $S^{\widehat{m}\widehat{n}}$ the full action is given by[3]

$$S = S_F + S_{scalar} + S_\rho + S_{int} \,, \tag{6.2.10}$$

[3]The numerical prefactors are determined by supersymmetry.

where

$$
\begin{aligned}
S_F &= -\frac{1}{4}\int \frac{1}{\ell(\theta)}\ \mathrm{Tr}(F\wedge *_{5d}F) \\
S_{\text{scalar}} &= +\frac{1}{16}\int d^5x\ \sqrt{|g|}\,\ell(\theta)\,\mathrm{Tr}\left(\Phi^{\widehat{m}\widehat{n}}\mathcal{D}^2\Phi_{\widehat{m}\widehat{n}} + \Phi^{\widehat{m}\widehat{n}}(M_\Phi)_{\widehat{m}\widehat{n}}^{\widehat{r}\widehat{s}}\Phi_{\widehat{r}\widehat{s}}\right) \\
S_\rho &= -\int d^5x\ \sqrt{|g|}\,\ell(\theta)\,\mathrm{Tr}\left(i\rho_{m'\widehat{m}}\not{\mathcal{D}}^{m'}_{n'}\rho^{n'\widehat{m}} + \rho_{m'\widehat{m}}(M_\rho)^{\widehat{m}\widehat{n}m'}{}_{n'}\,\rho^{n'}_{\widehat{n}}\right)\,.
\end{aligned}
\tag{6.2.11}
$$

Here, we non-abelianized the theory, and the covariant derivatives and mass matrices

$$
\begin{aligned}
\mathcal{D}_{\mu'}\Phi^{\widehat{m}\widehat{n}} &= \partial_{\mu'} + [A_{\mu'},\Phi^{\widehat{m}\widehat{n}}] \\
\mathcal{D}^2\Phi^{\widehat{m}\widehat{n}} &= \partial^{a'}\mathcal{D}_{a'}\Phi^{\widehat{m}\widehat{n}} + \frac{\ell'(\theta)}{r^2\ell(\theta)}\,\mathcal{D}_\theta\Phi^{\widehat{m}\widehat{n}} + [A_{\mu'},\partial^{\mu'}\Phi^{\widehat{m}\widehat{n}}] + [A_{\mu'},[A^{\mu'},\Phi^{\widehat{m}\widehat{n}}]] \\
D_{\mu'}\rho^{\widehat{m}} &= \partial_{\mu'}\rho^{\widehat{m}} + [A_{\mu'},\rho^{\widehat{m}}] \\
(M_\Phi)_{\widehat{r}\widehat{s}}^{\widehat{m}\widehat{n}} &= \frac{2\ell''(\theta)}{r^2\ell(\theta)}\,\delta^{\widehat{m}}_{[\widehat{r}}\delta^{\widehat{n}}_{\widehat{s}]} + \frac{1}{2\ell(\theta)^2}\left(S^{\widehat{m}}_{[\widehat{r}}S^{\widehat{n}}_{\widehat{s}]} - S^{\widehat{n}}_{\widehat{t}}S^{\widehat{t}}_{[\widehat{r}}\delta^{\widehat{m}}_{\widehat{s}]}\right) - \frac{1}{15}D^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}} \\
(M_\rho)^{\widehat{m}\widehat{n}m'}{}_{n'} &= \frac{1}{\ell(\theta)}\left(\frac{1}{2}S^{\widehat{m}\widehat{n}}\delta^{m'}_{n'} + \frac{i\ell'(\theta)}{2r}\,\Omega^{\widehat{m}\widehat{n}}(\gamma_5)^{m'}_{n'}\right)\,,
\end{aligned}
\tag{6.2.12}
$$

where the 5-dimensional Ricci scalar vanishes because we have a flat metric on the interval. In the non-abelian case we can add the following interaction terms

$$
\begin{aligned}
S_{\text{int}} = \int d^5x\sqrt{|g|}\mathrm{Tr}\Big(&\frac{\ell(\theta)^3}{64}[\Phi_{\widehat{m}\widehat{n}},\Phi^{\widehat{n}\widehat{r}}][\Phi_{\widehat{r}\widehat{s}},\Phi^{\widehat{s}\widehat{m}}] + \frac{\ell(\theta)}{24}S_{\widehat{m}\widehat{n}}\Phi^{\widehat{m}\widehat{r}}[\Phi^{\widehat{n}\widehat{s}},\Phi_{\widehat{r}\widehat{s}}] \\
&- \ell(\theta)^2\rho_{m'\widehat{m}}[\Phi^{\widehat{m}\widehat{n}},\rho^{m'}_{\widehat{n}}]\Big)\,,
\end{aligned}
\tag{6.2.13}
$$

where the non-vanishing background fields take the following values

$$
\begin{aligned}
S^{\widehat{m}}_{\widehat{n}} &= -r\ell'(\theta)(\Gamma^{\widehat{45}})^{\widehat{m}}_{\widehat{n}} \\
D^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}} &= \frac{3\ell''(\theta)}{2r^2\ell(\theta)}\left[5(\Gamma^{\widehat{45}})^{[\widehat{m}}_{\widehat{r}}(\Gamma^{\widehat{45}})^{\widehat{n}]}_{\widehat{s}} - \delta^{[\widehat{m}}_{\widehat{r}}\delta^{\widehat{n}]}_{\widehat{s}} - \Omega^{\widehat{m}\widehat{n}}\Omega_{\widehat{r}\widehat{s}}\right]\,,
\end{aligned}
\tag{6.2.14}
$$

where ℓ' and ℓ'' denote first and second derivatives of ℓ with respect to θ. The action is invariant under the following supersymmetry transformations

$$
\begin{aligned}
\delta A_{\mu'} &= \ell(\theta)\,\epsilon_{\widehat{m}}\gamma_{\mu'}\rho^{\widehat{m}} \\
\delta\Phi^{\widehat{m}\widehat{n}} &= -4i\epsilon^{[\widehat{m}}\rho^{\widehat{n}]} \\
\delta\rho^{\widehat{m}} &= \frac{i}{8\ell(\theta)}F_{\mu'\nu'}\gamma^{\mu'\nu'}\epsilon^{\widehat{m}} + \frac{1}{4}\not{\mathcal{D}}\Phi^{\widehat{m}\widehat{n}}\epsilon_{\widehat{n}} + \frac{i}{4\ell(\theta)}S^{[\widehat{m}}_{\widehat{r}}\Phi^{\widehat{n}]\widehat{r}}\epsilon_{\widehat{n}} - \frac{i}{8}\ell(\theta)\Omega_{\widehat{n}\widehat{r}}[\Phi^{\widehat{m}\widehat{n}},\Phi^{\widehat{r}\widehat{s}}]\epsilon_{\widehat{s}}\,.
\end{aligned}
\tag{6.2.15}
$$

Note that the Killing spinor $\epsilon^{m'}_{\widehat{m}}$ satisfies the relation (6.1.25) which now reads

$$
(\Gamma^{\widehat{45}})^{\widehat{m}\widehat{n}}\epsilon^{m'}_{\widehat{n}} = -i(\gamma_5)^{m'}_{n'}\,\epsilon^{n'\widehat{m}}\,. \tag{6.2.16}
$$

So far we have kept the $\mathfrak{sp}(4)_R$ R-symmetry indices explicit. However the background breaks the R-symmetry to $\mathfrak{su}(2)_R\oplus\mathfrak{so}(2)_R$. To make the symmetry of the theory manifest, we decompose the scalar fields $\Phi^{\widehat{m}\widehat{n}}$ into a triplet of scalars $\varphi^{\widehat{a}}$, transforming in the $\mathbf{3_0}$ of $\mathfrak{su}(2)_R \oplus \mathfrak{so}(2)_R$, and the complex field φ, which is singlet $\mathbf{1_1}$, which can be achieved as follows

$$\begin{aligned} \varphi^{\widehat{a}} &= \frac{1}{4}(\Gamma^{\widehat{a}})_{\widehat{m}\widehat{n}}\Phi^{\widehat{m}\widehat{n}}\,, \quad \widehat{a}=1,2,3 \\ \varphi = \varphi^4 + i\varphi^5 &= \frac{1}{4}\left(\Gamma^4 + i\Gamma^5\right)_{\widehat{m}\widehat{n}}\Phi^{\widehat{m}\widehat{n}}\,. \end{aligned} \tag{6.2.17}$$

The spinors $\rho_{\widehat{m}}$ decompose into the two doublets $\rho^{(1)}_{\widehat{p}}$, $\rho^{(2)}_{\widehat{p}}$, transform in the $(\mathbf{2})_{\mathbf{1}}\oplus(\mathbf{2})_{\mathbf{-1}}$, as detailed in appendix B.1.3. We also split the gauge field (singlet of the R-symmetry) into the components A_μ along $\mathbb{R}^4$ and the component A_θ along the interval.

The spinor $\epsilon_{\widehat{n}}$ parametrizing supersymmetry transformations decompose under the R-symmetry subalgebra $\mathfrak{su}(2)_R \oplus \mathfrak{so}(2)_R$ into two $\mathfrak{su}(2)_R$ doublets of opposite $\mathfrak{so}(2)_R$ charge: $\epsilon_{\widehat{m}} \to \epsilon^{(1)}_{\widehat{p}} \oplus \epsilon^{(2)}_{\widehat{p}}$ (see appendix B.1.3). The projection condition (6.2.16) becomes

$$\epsilon^{(1)}_{\widehat{p}} - \gamma^5\epsilon^{(1)}_{\widehat{p}} = 0\,, \quad \epsilon^{(2)}_{\widehat{p}} + \gamma^5\epsilon^{(2)}_{\widehat{p}} = 0\,. \tag{6.2.18}$$

For any 5-dimensional spinor χ we define

$$\chi_\pm = \frac{1}{2}(\chi \pm \gamma^5\chi), \tag{6.2.19}$$

as the 4-dimensional chirality. The action for the gauge field is

$$S_F = -\frac{1}{8}\int d^5x\sqrt{|g|}\;\frac{1}{\ell(\theta)}\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\theta}F^{\mu\theta}\right)\,. \tag{6.2.20}$$

The action for the scalars is

$$\begin{aligned} &S_{\text{scalar}} \\ &= -\frac{1}{4}\int d^5x\;\ell(\theta)\mathrm{Tr}\left(\mathcal{D}^{\mu'}\varphi^{\hat{a}}\mathcal{D}_{\mu'}\varphi_{\hat{a}} + \mathcal{D}^{\mu'}\varphi\mathcal{D}_{\mu'}\bar{\varphi} + \frac{1}{r^2}D_\theta\varphi^{\widehat{a}}D_\theta\varphi_{\widehat{a}} + \frac{1}{r^2}D_\theta\varphi D_\theta\bar{\varphi} + m_\varphi^2\varphi\bar{\varphi}\right)\,, \end{aligned} \tag{6.2.21}$$

where the mass term is

$$m_\varphi(\theta)^2 = \frac{\ell'(\theta)^2 - \ell(\theta)\ell''(\theta)}{r^2\ell(\theta)^2}\,. \tag{6.2.22}$$

which for the round sphere reduces to $m_\varphi^2 = \cot(\theta)^2/r^2$ and diverges at the endpoints of the interval. We will return to this matter when discussing the boundary conditions. The action for the fermions is

$$\begin{aligned} \mathrm{S}_\rho = -2i\int d^5x\sqrt{|g|}\;\ell(\theta)\,\mathrm{Tr}\,\big(&\rho^{(1)}_{\hat{p}+}\gamma^\mu D_\mu\rho^{(2)\hat{p}}_{-} + \rho^{(1)}_{\hat{p}-}\gamma^\mu D_\mu\rho^{(2)\hat{p}}_{+} \\ &+ \frac{1}{r}\rho^{(1)}_{\hat{p}+}\mathcal{D}_\theta\rho^{(2)\hat{p}}_{+} - \frac{1}{r}\rho^{(2)}_{\hat{p}-}\mathcal{D}_\theta\rho^{(1)\hat{p}}_{-}\big)\,, \end{aligned} \tag{6.2.23}$$

and the interaction terms become

$$
\begin{aligned}
S_{\text{Yukawa}} &= -\int d^5x\sqrt{|g|}\,\ell(\theta)^2\,\text{Tr}\Big[2(\sigma^{\hat a})^{\hat p\hat q}\rho^{(2)}_{\hat p-}\left[\varphi_{\hat a},\rho^{(1)}_{\hat q-}\right]+2(\sigma^{\hat a})^{\hat p\hat q}\rho^{(2)}_{\hat p+}\left[\varphi_{\hat a},\rho^{(1)}_{\hat q+}\right]\\
&\qquad +i\left(\rho^{(1)}_{\hat p-}\left[\bar\varphi,\rho^{\hat p(1)}_{-}\right]+\rho^{(1)}_{\hat p+}\left[\bar\varphi,\rho^{\hat p(1)}_{+}\right]-\rho^{(2)}_{\hat p-}\left[\varphi,\rho^{\hat p(2)}_{-}\right]-\rho^{(2)}_{\hat p+}\left[\varphi,\rho^{\hat p(2)}_{+}\right]\right)\Big]\\
S_{\text{quartic}} &= -\frac{1}{4}\int d^5x\sqrt{|g|}\,\ell(\theta)^3\,\text{Tr}\left([\varphi_{\hat a},\varphi][\varphi^{\hat a},\bar\varphi]+\frac{1}{2}[\varphi_{\hat a},\varphi_{\hat b}][\varphi^{\hat a},\varphi^{\hat b}]-\frac{1}{4}[\varphi,\bar\varphi][\varphi,\bar\varphi]\right)\\
S_{\text{cubic}} &= \frac{1}{6}\int d^5x\sqrt{|g|}\,\frac{\ell(\theta)\ell'(\theta)}{r}\,\epsilon^{\hat a\hat b\hat c}\,\text{Tr}\left(\varphi_{\hat a}[\varphi_{\hat b},\varphi_{\hat c}]\right)\,.
\end{aligned}
\tag{6.2.24}
$$

The complete 5-dimensional action is

$$
S_{5d}=S_F+S_{\text{scalar}}+S_\rho+S_{\text{Yukawa}}+S_{\text{quartic}}+S_{\text{cubic}},
\tag{6.2.25}
$$

and the supersymmetry variations for this action, decomposed with regards to the R-symmetry, are summarized in appendix B.4. The action above should be supplemented with appropriate boundary terms, which ensure that supersymmetry is preserved and that the action is finite. We will address this issue after we have taken the cylinder limit of the metric.

We need to determine the boundary conditions of the 5-dimensional fields at the endpoints of the θ interval.

To proceed we first notice that the complex scalar φ has a mass term $m(\theta)^2$ which diverges at the boundaries $\theta=0,\pi$: [4]

$$
m(\theta)^2\simeq\begin{cases}\frac{1}{\theta^2} & ,\quad \theta\to 0\,,\\ \frac{1}{(\pi-\theta)^2} & ,\quad \theta\to\pi\,.\end{cases}
\tag{6.2.26}
$$

Finiteness of the action requires that φ behaves as

$$
\varphi=\begin{cases}O(\theta) & ,\quad \theta\to 0\,,\\ O(\pi-\theta) & ,\quad \theta\to\pi\,.\end{cases}
\tag{6.2.27}
$$

The boundary conditions on the other fields are most easily determined by the requirement to preserve supersymmetry under the transformations generated by $\epsilon^{(1)}_{\widehat p}$ and $\epsilon^{(2)}_{\widehat p}$ presented in appendix B.4. We obtain at $\theta=0$:

$$
\begin{aligned}
&\rho^{(1)}_{\widehat p+}=O(\theta)\,,\quad \rho^{(2)}_{\widehat p-}=O(\theta)\,,\\
&A_\mu=O(\theta^2)\,,
\end{aligned}
\tag{6.2.28}
$$

and the counterpart at $\theta=\pi$.

[4]This follows from the regularity conditions on f: $f(\theta)\simeq\theta$ at $\theta=0$ and $f(\theta)\simeq\pi-\theta$ at $\theta=\pi$.

The fields $\varphi^{\widehat{a}}, A_\theta$ are constrained to obey generalized Nahm equations as they approach the boundaries. The generalized Nahm equations take the form

$$\mathcal{D}_\theta \varphi^{\widehat{a}} - \frac{1}{2} r\ell(\theta) \epsilon^{\widehat{a}}{}_{\widehat{b}\widehat{c}} \varphi^{\widehat{b}} \varphi^{\widehat{c}} = 0 \,. \tag{6.2.29}$$

These equations reduce to standard Nahm's equations upon setting $r\ell(\theta) = 1$[5].These equations are compatible with a singular boundary behaviour of the fields at the endpoints of the θ interval. For simplicity let us assume the gauge $A_\theta = 0$ in a neighbourhood of $\theta = 0$, then the above modified Nahm's equation are compatible with the polar behaviour at $\theta = 0$

$$\varphi^{\widehat{a}} = \frac{2\varrho(\tau^{\widehat{a}})}{r\theta^2} + O(1) \,. \tag{6.2.30}$$

where

$$\varrho : \mathfrak{su}(2) \to \mathfrak{u}(k) \tag{6.2.31}$$

denotes an embedding of $\mathfrak{su}(2)$ into $\mathfrak{u}(k)$, see e.g. in ([140,164]) and $\tau^{\widehat{a}}$ are related to the Pauli matrices $\sigma^{\widehat{a}}$ as follows

$$\tau^{\widehat{a}} = \frac{i}{2} \sigma^{\widehat{a}} \,. \tag{6.2.32}$$

Moreover the order $O(1)$ term is constrained to be in the commutant of ϱ in $\mathfrak{u}(k)$. The reduction that we study, from a smooth two-sphere to the interval, correspond to ϱ being an irreducible embedding ([140]).

More generaly the Nahm pole boundary condition (6.2.29) is compatible with any embedding ϱ and is associated with the presence of 'punctures' – or field singularities – at the poles of the two-sphere in the 6-dimensional non-abelian theory ([133]). An embedding ϱ can be associated to a decomposition of the fundamental representation (k) under $\mathfrak{su}(2)$ and can be recast into a partition $[n_1, n_2, \cdots]$ of k. The irreducible embedding is associated to the partition $\varrho = [k]$ and corresponds to the absence of puncture in 6d. The boundary conditions at $\theta = \pi$ are the mirror of the one at $\theta = 0$ and are also characterized by Nahm pole behaviour with irreducible embedding $\varrho = [k]$.

The remaining fermions $\rho_-^{(1)}, \rho_+^{(2)}$ appear in the supersymmetry variations of $\varphi^{\widehat{a}}$ and hence are of order $O(1)$ at $\theta = 0$

$$\rho_{\widehat{p}-}^{(1)} = O(1) \,, \quad \rho_{\widehat{p}+}^{(2)} = O(1) \,, \tag{6.2.33}$$

and similarly at $\theta = \pi$.

[5] After a redefinition of θ, these equations are simply a different form of the usual Nahm's equations.

6.2.3 Cylinder Limit

For general hyperbolic Riemann surfaces, the dimensional reduction depends only on the complex structure moduli ([133]). The two-sphere has no complex structure moduli, however, there will be a metric-dependence in terms of the area of the sphere, which enters as the function $\ell(\theta)$, except through the area of the sphere. This can be checked by explicitly performing the reduction keeping $\ell(\theta)$ arbitrary. However, for simplicity we consider here the special limiting case, when the two-sphere is deformed to a thin cylinder capped with two half-spheres. This corresponds to taking the metric factor $\ell(\theta)$ to be be constant

$$\begin{aligned} &\ell(\theta) = \ell = \text{const.} \quad \text{for } \epsilon < \theta < \pi - \epsilon\,, \\ &\ell(\theta) \to \text{ smooth caps} \quad \text{for } \theta < \epsilon\,,\ \pi - \epsilon < \theta\,. \end{aligned} \tag{6.2.34}$$

and taking the limit $\epsilon \to 0$. The limit is singular at the endpoints of the θ-interval. The boundary conditions in 5 dimensions can then be fixed by preserving supersymmetry and the symmetry group of the twisted theory. We rescale the fields as follows

$$\varphi^{\hat{a}} \to \frac{1}{r\ell}\varphi^{\hat{a}}\,, \qquad \varphi \to \frac{1}{r\ell}\varphi\,, \qquad \rho_\pm^{(1)} \to \frac{1}{r\ell}\rho_\pm^{(1)}\,, \qquad \rho_\pm^{(2)} \to \frac{1}{r\ell}\rho_\pm^{(2)}\,. \tag{6.2.35}$$

The action in this limit simplifies to

$$\begin{aligned} S_F &= -\frac{r}{8\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu} + \frac{2}{r^2}(\partial_\mu A_\theta - \partial_\theta A_\mu + [A_\mu, A_\theta])^2\right) \\ S_{\text{scalar}} &= -\frac{1}{4r\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\left(\mathcal{D}^{\mu'}\varphi^{\hat{a}}\mathcal{D}_{\mu'}\varphi_{\hat{a}} + \mathcal{D}^{\mu'}\varphi\mathcal{D}_{\mu'}\bar{\varphi}\right) \\ S_\rho &= -\frac{2i}{r\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\Big(\rho^{(1)}_{\hat{p}+}\gamma^\mu\mathcal{D}_\mu\rho_-^{(2)\hat{p}} + \rho^{(1)}_{\hat{p}-}\gamma^\mu\mathcal{D}_\mu\rho_+^{(2)\hat{p}} \\ &\qquad + \frac{1}{r}\rho^{(1)}_{\hat{p}+}\mathcal{D}_\theta\rho_+^{(2)\hat{p}} - \frac{1}{r}\rho^{(1)}_{\hat{p}-}\mathcal{D}_\theta\rho_-^{(2)\hat{p}}\Big) \\ S_{\text{Yukawa}} &= -\frac{1}{r^2\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\Big(2\rho^{(2)}_{\hat{p}-}\left[\varphi^{\widehat{pq}}, \rho^{(1)}_{\hat{q}-}\right] + 2\rho^{(2)}_{\hat{p}+}\left[\varphi^{\widehat{pq}}, \rho^{(1)}_{\hat{q}+}\right] \\ &\qquad + i\left(\rho^{(1)}_{\hat{p}-}\left[\bar{\varphi}, \rho_-^{\hat{p}(1)}\right] + \rho^{(1)}_{\hat{p}+}\left[\bar{\varphi}, \rho_+^{\hat{p}(1)}\right] - \rho^{(2)}_{\hat{p}-}\left[\varphi, \rho_-^{\hat{p}(2)}\right] - \rho^{(2)}_{\hat{p}+}\left[\varphi, \rho_+^{\hat{p}(2)}\right]\right)\Big) \\ S_{\text{quartic}} &= -\frac{1}{4r^3\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\left(\frac{1}{2}[\varphi_{\hat{a}}, \varphi_{\hat{b}}][\varphi^{\hat{a}}, \varphi^{\hat{b}}] + [\varphi_{\hat{a}}, \varphi][\varphi^{\hat{a}}, \bar{\varphi}] - \frac{1}{4}[\varphi, \bar{\varphi}][\varphi, \bar{\varphi}]\right)\,. \end{aligned} \tag{6.2.36}$$

The supersymmetry variations in appendix B.4 in the cylinder limit reduce to

$$\begin{aligned}
\delta A_\mu &= -\frac{1}{r}\left(\epsilon^{(1)\widehat{p}}\gamma_\mu\rho^{(2)}_{\widehat{p}-} + \epsilon^{(2)\widehat{p}}\gamma_\mu\rho^{(1)}_{\widehat{p}+}\right)\\
\delta A_\theta &= -\left(\epsilon^{(1)\widehat{p}}\rho^{(2)}_{\widehat{p}+} - \epsilon^{(2)\widehat{p}}\rho^{(1)}_{\widehat{p}-}\right)\\
\delta\varphi^{\widehat{a}} &= i\left(\epsilon^{(1)}{}_{\widehat{p}}(\sigma^{\widehat{a}})^{\widehat{p}\widehat{q}}\rho^{(2)}_{\widehat{q}+} - \epsilon^{(2)}{}_{\widehat{p}}(\sigma^{\widehat{a}})^{\widehat{p}\widehat{q}}\rho^{(1)}_{\widehat{q}-}\right)\\
\delta\varphi &= -2\epsilon^{(1)\widehat{p}}\rho^{(1)}_{\widehat{p}+}\\
\delta\bar{\varphi} &= +2\epsilon^{(2)\widehat{p}}\rho^{(2)}_{\widehat{p}-}
\end{aligned} \tag{6.2.37}$$

for the bosonic fields and for the fermions

$$\begin{aligned}
\delta\rho^{(1)}_{\widehat{p}+} &= \frac{ir}{8}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^{(1)}_{\widehat{p}} - \frac{i}{4}\mathcal{D}_\mu\varphi\gamma^\mu\epsilon^{(2)}_{\widehat{p}} + \frac{1}{4r}\mathcal{D}_\theta\varphi^{\widehat{q}}_{\widehat{p}}\epsilon^{(1)}_{\widehat{q}} - \frac{1}{8r}\left(\epsilon^{\widehat{a}\widehat{b}\widehat{c}}[\varphi_{\widehat{a}},\varphi_{\widehat{b}}](\sigma_{\widehat{c}})^{\widehat{q}}_{\widehat{p}}\epsilon^{(1)}_{\widehat{q}} - i[\varphi,\bar{\varphi}]\epsilon^{(1)}_{\widehat{p}}\right)\\
\delta\rho^{(1)}_{\widehat{p}-} &= \frac{i}{4}F_{\mu\theta}\gamma^\mu\epsilon^{(1)}_{\widehat{p}} + \frac{1}{4}\mathcal{D}_\mu\varphi^{\widehat{q}}_{\widehat{p}}\gamma^\mu\epsilon^{(1)}_{\widehat{q}} + \frac{i}{4r}\mathcal{D}_\theta\varphi\epsilon^{(2)}_{\widehat{p}} - \frac{1}{4r}[\varphi,\varphi^{\widehat{q}}_{\widehat{p}}]\epsilon^{(2)}_{\widehat{q}}\\
\delta\rho^{(2)}_{\widehat{p}+} &= -\frac{i}{4}F_{\mu\theta}\gamma^\mu\epsilon^{(2)}_{\widehat{p}} - \frac{1}{4}\mathcal{D}_\mu\varphi^{\widehat{q}}_{\widehat{p}}\gamma^\mu\epsilon^{(2)}_{\widehat{q}} - \frac{i}{4r}\mathcal{D}_\theta\bar{\varphi}\epsilon^{(1)}_{\widehat{p}} - \frac{1}{4r}[\bar{\varphi},\varphi^{\widehat{q}}_{\widehat{p}}]\epsilon^{(1)}_{\widehat{q}}\\
\delta\rho^{(2)}_{\widehat{p}-} &= \frac{ir}{8}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^{(2)}_{\widehat{p}} + \frac{i}{4}\mathcal{D}_\mu\bar{\varphi}\gamma^\mu\epsilon^{(1)}_{\widehat{p}} + \frac{1}{4r}\mathcal{D}_\theta\varphi^{\widehat{q}}_{\widehat{p}}\epsilon^{(2)}_{\widehat{q}} - \frac{1}{8r}\left(\epsilon^{\widehat{a}\widehat{b}\widehat{c}}[\varphi_{\widehat{a}},\varphi_{\widehat{b}}](\sigma_{\widehat{c}})^{\widehat{q}}_{\widehat{p}}\epsilon^{(2)}_{\widehat{q}} + i[\varphi,\bar{\varphi}]\epsilon^{(2)}_{\widehat{p}}\right).
\end{aligned} \tag{6.2.38}$$

The theory we obtain is nothing else than the $\mathcal{N}=2$ Super-Yang-Mills theory in 5d. A similar reduction of the 6-dimensional (0,2) theory on a cigar geometry was considered in ([164]). The 5-dimensional Super-Yang-Mills theory is defined on a manifold with boundaries, which are at the end-points of the θ-interval and half of the supersymmetries are broken by the boundary conditions. It is key to study the boundary terms and boundary conditions in detail, which will be done in the next subsection.

6.2.4 Nahm Equations and Boundary Considerations

The boundary conditions at the two ends of the θ interval can be worked out in the same way as in section 6.2.2. In the cylinder limit of the two-sphere $\ell(\theta)\to\ell$ the mass term $m(\theta)^2$ goes to zero everywhere on the θ interval except at the endpoints $\theta=0,\pi$ where is diverges, forcing the scalar φ to vanish at the boundary. The other boundary conditions can be worked out by requiring supersymmetry under the eight supercharges. This leads to letting the scalars $\varphi^{\widehat{a}}$ obey standard Nahm equations close to the boundaries

$$\mathcal{D}_\theta\varphi^{\hat{a}} - \frac{1}{2}\epsilon^{\widehat{a}}{}_{\widehat{b}\widehat{c}}[\varphi^{\widehat{b}},\varphi^{\widehat{c}}] = 0\,. \tag{6.2.39}$$

Explicitly we obtain, in the gauge $A_\theta=0$, at $\theta=0$:

$$
\begin{aligned}
&\varphi = O(\theta)\,, \quad A_\mu = O(\theta)\,,\\
&\varphi^{\widehat{a}} = \frac{\varrho(\tau^{\widehat{a}})}{\theta} + \varphi^{\widehat{a}}_{(0)} + O(\theta)\,,\\
&\rho^{(1)}_{\widehat{p}-} = O(1)\,, \quad \rho^{(2)}_{\widehat{p}+} = O(1)\,,\\
&\rho^{(1)}_{\widehat{p}+} = O(\theta)\,, \quad \rho^{(2)}_{\widehat{p}-} = O(\theta)\,,
\end{aligned}
\tag{6.2.40}
$$

where $\varrho : \mathfrak{su}(2) \to \mathfrak{u}(k)$ is an irreducible embedding of $\mathfrak{su}(2)$ into $\mathfrak{u}(k)$, with τ as in (6.2.32) and there are identical boundary conditions at $\theta = \pi$. The constant term $\varphi^{\widehat{a}}_{(0)}$ in the $\varphi^{\widehat{a}}$-expansion is constrained to be in the commutant of the embedding ϱ. The maximally supersymmetric configurations are vacua of the theory preserving the eight supercharges and are given by the BPS equations

$$
\begin{aligned}
\mathcal{D}_\theta\varphi^{\hat{a}} - \frac{1}{2}\epsilon^{\widehat{a}}{}_{\widehat{b}\widehat{c}}[\varphi^{\widehat{b}}, \varphi^{\widehat{c}}] &= 0\\
\varphi = \bar{\varphi} = F_{\mu\nu} = F_{\mu\theta} &= 0\\
\mathcal{D}_\mu\varphi_{\hat{a}} &= 0\,.
\end{aligned}
\tag{6.2.41}
$$

and vanishing fermions. Moreover there is the additional constraint that the scalars $\varphi^{\hat{a}}$ have poles at $\theta = 0, \pi$ both characterized by the partition $\varrho = [k]$. The first equation in (6.2.41) is the Nahm equation for the fields $(\varphi^{\widehat{a}}, A_\theta)$ and the boundary behaviour of $\varphi^{\widehat{a}}$ are standard Nahm poles.

The Nahm pole boundary condition introduces two difficulties: the supersymmetry variation of the action results in a non-vanishing boundary term and the polar behaviour of the scalar fields make the action diverge. These two problems are cured by the addition of the following boundary term

$$
\begin{aligned}
S_{\text{bdry}} &= -\frac{1}{6r^2\ell}\text{Tr}\int d^4x\sqrt{|g|}\left[\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\varphi^{\widehat{a}}\varphi^{\widehat{b}}\varphi^{\widehat{c}}\right]_0^\pi\\
&= -\frac{1}{6r^2\ell}\text{Tr}\int d^4x d\theta\sqrt{|g|}\partial_\theta\left(\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\varphi^{\widehat{a}}\varphi^{\widehat{b}}\varphi^{\widehat{c}}\right)\,.
\end{aligned}
\tag{6.2.42}
$$

The second line gives S_{bdry} a total θ-derivative and we shall take this as the definition of S_{bdry}. This additional term ensures supersymmetry and makes the 5-dimensional action finite. We have in particular

$$
\begin{aligned}
-\text{Tr}\int d\theta d^4x\sqrt{|g|}\left(\mathcal{D}_\theta\varphi^{\widehat{a}}\mathcal{D}_\theta\varphi_{\widehat{a}} + \frac{1}{2}[\varphi_{\widehat{a}}, \varphi_{\widehat{b}}][\varphi^{\widehat{a}}, \varphi^{\widehat{b}}]\right) - \frac{2}{3}\text{Tr}\int d^4x d\theta\sqrt{|g|}\partial_\theta\left(\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\varphi^{\widehat{a}}\varphi^{\widehat{b}}\varphi^{\widehat{c}}\right)\\
= \text{Tr}\int d^4x d\theta\sqrt{|g|}\left(D_\theta\varphi_{\widehat{a}} - \frac{1}{2}\epsilon_{\widehat{a}\widehat{b}\widehat{c}}[\varphi^{\widehat{b}}, \varphi^{\widehat{c}}]\right)^2\,.
\end{aligned}
\tag{6.2.43}
$$

which is the square of Nahm's equations. The 5-dimensional action is finite since the scalar fields $\varphi^{\widehat{a}}$ obey Nahm's equations at the boundaries. We even find a stronger result: the 5-dimensional action is minimized when the scalars $\varphi^{\widehat{a}}$ obey Nahm's equations on the whole θ interval.

Finally we can comment of generalizations of the Nahm pole boundary conditions to two arbitrary partition ϱ_o and ϱ_π for the scalars fields at the two boundaries $\theta = 0, \pi$ as described in ([140]). These boundary conditions preserve the same amount of supersymmetry and admit global symmetry groups $H_0 \times H_\pi \subset SU(k) \times SU(k)$ acting by global gauge transformations on the fields. H_0 and H_π are the commutant of ϱ_o and ϱ_π in $SU(Nk)$, defined as the subgroup of $SU(k)$ which leaves the ϱ boundary conditions invariant. The general (ϱ_0, ϱ_π) Nahm pole boundary condition corresponds to inserting singularities or punctures at the two poles of the two-sphere in the 6-dimensional (2,0) theory (see ([140])). All our results can be directly generalized to having general Nahm poles at the boundaries of the θ interval.

6.3 4-dimensional Sigma-Model and Nahm's Equations

In the last section we have seen that the 5-dimensiona Super-Yang-Mills theory in the background corresponding to the S^2 reduction of the 6-dimensional $(2,0)$ theory requires the scalars $\varphi^{\hat{a}}$ to satisfy Nahm's equations, and the supersymmetric boundary conditions require them to have Nahm poles (6.2.40) at the boundaries of the interval. The 4-dimensional theory is therefore dependent on solutions to Nahm's equations. To dimensionally reduce the theory, we pass to a description in terms of of coordinates on the moduli space $\mathcal{M}_k$ of solutions to Nahm's equations. In this section we will find the theory to be a 4-dimensional sigma model into $\mathcal{M}_k$, where the bosonic degrees of freedom X^I, $I = 1, 2, \ldots, 4k$, are coordinates on the moduli space

$$X^I : M_4 \to \mathcal{M}_k \,, \tag{6.3.1}$$

while the fermionic degrees of freedom $\xi^{(i)}$, $i = 1, 2$, are Grassman-valued sections of the pull-back of the tangent bundle to $\mathcal{M}$

$$\xi^{(1,2)} \in \Gamma(X^* T\mathcal{M} \otimes \mathcal{S}^{\pm}) \,, \tag{6.3.2}$$

where $\mathcal{S}^{\pm}$ is the spin bundle on M_4. The sigma-model for $M_4 = \mathbb{R}^4$ is supersymmetric, with $\mathcal{N} = 2$ supersymmetry in 4 dimensions. The coupling constant for the sigma-model is proportional to the area of the two-sphere as anticipated. We now proceed to the derivation of the sigma model.

6.3.1 Poles and Monopoles

Before studying the dimensional reduction to 4 dimensions, we summarize a few well-known useful properties of the moduli space $\mathcal{M}_k$. The moduli space $\mathcal{M}_k$ of solutions to Nahm's equations, on an interval with Nahm pole boundary conditions given by the irreducible embedding $\rho = [k]$, is well-known to be isomorphic to the moduli space of (framed) $\mathfrak{su}(2)$ magnetic monopoles of charge k ([165, 166]), which is $4k$-dimensional and has a Hyper-Kahler structure. Let us recall the monopole moduli space and the Nahm equations moduli space.

Magnetic monopoles can be understood as solutions to the Bogomolny equations ([167]). Let A_μ be a connection on a principal G-bundle over a 3-dimensional manifold M_3. Moreover, let ϕ be a section of the associated adjoint bundle, the so-called Higgs field. The Bogomolny equation for the pair (A, ϕ) is then ([167])

$$F = \star D\phi, \tag{6.3.3}$$

where F is the field strength of the connection, D is the covariant derivative defined by A_μ and $\star$ is the 3-dimensional Hodge operator. Moreover, it can be proved that the following action ([166])

$$S = \frac{1}{2} \int_{M_3} (F, F) + (D\phi, D\phi), \tag{6.3.4}$$

where $(\, , \,)$ is the Lorentz invariant inner product, is minimised when the couple (A, ϕ) satisfies the Bogomolny equations. In addition, in that case ([166]), $S = 4\pi k$, where k is then defined as the monopole magnetic charge. The solutions to such Bogomolny equations describe the monopole moduli space of charge k.

On the other hand, as we saw, Nahm equations read as follows

$$\frac{dT_i}{d\theta} - \frac{1}{2}\epsilon_{ijk}[T_j, T_k] = 0, \quad i = 1, 2, 3, \tag{6.3.5}$$

where T_i are matrix valued, depending on $\theta \in [0, \pi]$ and have simple poles at the endpoints of the interval, the residues of which define representations of $\mathfrak{su}(2)$. In particular the residue is specified by an embedding

$$\rho : \mathfrak{su}(2) \to \mathfrak{u}(k), \tag{6.3.6}$$

which is determined by a partition $[k]$ of k. This is because such embedding is specified by the image of the nilpotent matrix $\sigma^+ = \sigma^1 + i\sigma^2$ in $\mathfrak{u}(k)$ (where σ_i are the Pauli matrices), which corresponds to a partition of k following the decomposition into Jordan normal form (see e.g. ([168])).

Hitchin showed the equivalence of the moduli space of $\mathfrak{su}(2)$ monopoles of charge k with the moduli space of Nahm's equations with boundary conditions specified by the embedding $\rho = [k]$ ([153]).

The metric of spaces $\mathcal{M}_k$ is not known in explicit form, other than for the cases $\mathcal{M}_1 \simeq \mathbb{R}^3 \times S^1$ and for the case

$$\mathcal{M}_2 \simeq \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_{AH}}{\mathbb{Z}_2}, \tag{6.3.7}$$

where $\mathcal{M}_{AH}$ is the Atiyah-Hitchin manifold ([166]). For general k the moduli space takes the form ([166])

$$\mathcal{M}_k \simeq \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_k^0}{\mathbb{Z}_k}. \tag{6.3.8}$$

A particularly useful characterization of the reduced Nahm moduli space $\mathcal{M}_k^0$ is in terms of Slodowy-slices. Bielawski showed in ([169, 170]), that the moduli space of solutions with Nahm pole boundary conditions for k-centered $\mathfrak{su}(2)$ monopoles is given in terms of

$$\mathcal{M}_k^0 \simeq \{(g, X) \in \mathfrak{su}(N)_{\mathbb{C}} \times \mathfrak{su}(N)_{\mathbb{C}}; X \in S_{[k]} \cup g^{-1} S_{[k]} g \subset T^* SU(k)_{\mathbb{C}}\} \tag{6.3.9}$$

where the Slodowy slice for an embedding $\rho : \mathfrak{su}(2) \to \mathfrak{u}(k)$ is

$$S_\rho = \{\rho(\tau^+) + x \in \mathfrak{su}(k)_{\mathbb{C}};\ [\rho(\tau^-), x] = 0\}. \tag{6.3.10}$$

Here $\sigma^{\pm} \equiv \sigma^1 \pm i\sigma^2$ are the raising/lowering operators of su(2).

6.3.2 Reduction to the 4-dimensional Sigma-Model

To proceed with the reduction on the θ interval to 4 dimensions, we take the limit of small r (the size of the interval). The terms in the action (6.2.36) are organized in powers of r. The divergent terms which are of order r^{-n}, $n = 2, 3$, must vanish separately. The terms of order r^{-1} contain the four-dimensional kinetic terms and lead to the 4-dimensional action. The terms of order r^n, $n \geq 0$ are subleading and can be set to zero. To perform this reduction we must expand a generic field Δ in powers of r,

$$\Delta = \Delta_0 + \Delta_1 r + \Delta_2 r^2 + \dots, \tag{6.3.11}$$

and compute the contribution at each order. We find that only the leading term Δ_0 contributes to the final 4-dimensional action for each field, except for the 'massive' scalars $\varphi, \bar{\varphi}$, and spinors $\rho^{(1)}_{+\widehat{p}}, \rho^{(2)}_{-\widehat{p}}$ whose leading contribution arise at order r. The final 4-dimensional action will arise with the overall coupling $\frac{1}{r\ell}$.

Let us now proceed with detailing the dimensional reduction. We substitute the expansion (6.3.11) in the action (6.2.36) and study the terms at each order in powers of r. At order r^{-3} we find the term

$$S_{r^{-3}} = -\frac{1}{4r^3\ell}\int d\theta d^4x\sqrt{|g|}$$
$$\mathrm{Tr}\left[\left(D_\theta\varphi^{\widehat{a}} - \frac{1}{2}\epsilon^{\widehat{a}}{}_{\widehat{b}\widehat{c}}[\varphi^{\widehat{b}},\varphi^{\widehat{c}}]\right)^2 + [\varphi_{\widehat{a}},\varphi][\varphi^{\widehat{a}},\bar{\varphi}] + D_\theta\varphi D_\theta\bar{\varphi} - \frac{1}{4}[\varphi,\bar{\varphi}][\varphi,\bar{\varphi}]\right] . \tag{6.3.12}$$

This term is minimized (and actually vanishes) up to order $O(r^{-1})$ corrections, upon imposing the following constraints: $\varphi,\bar{\varphi}$ vanish at order r^0,

$$\varphi = \bar{\varphi} = O(r), \tag{6.3.13}$$

and the fields $\varphi^{\widehat{a}}$ and A_θ obey Nahm's equations, up to order $O(r)$ corrections,

$$\mathcal{D}_\theta\varphi^{\hat{a}} - \frac{1}{2}\epsilon^{\widehat{a}}{}_{\widehat{b}\widehat{c}}[\varphi^{\widehat{b}},\varphi^{\widehat{c}}] = 0\,, \tag{6.3.14}$$

with Nahm pole behaviour $\varrho = [k]$ at the two ends of the interval. The 4-dimensional theory will then localize onto maps $X : \mathbb{R}^4 \to \mathcal{M}_k$, where $\mathcal{M}_k$ is the moduli space of $\mathfrak{u}(k)$ valued solutions of Nahm's equations on the interval with ϱ-poles at the boundaries. Furthermore we choose the gauge fixing

$$\partial_\theta A_\theta = 0. \tag{6.3.15}$$

The terms at $O(r^{-2})$ vanish by imposing $\rho^{(1)}_{\widehat{p}+},\rho^{(2)}_{\widehat{p}-}$ to have no $O(r^0)$ term

$$\rho^{(1)}_{\widehat{p}+} = O(r), \quad \rho^{(2)}_{\widehat{p}-} = O(r). \tag{6.3.16}$$

The kinetic term of these spinors become of order r and can be dropped in the small r limit. The fermions $\rho^{(1)}_{\widehat{p}+},\rho^{(2)}_{\widehat{p}-}$ become Lagrange multipliers and can then be integrated out, leading to the constraints on the fermions $\rho^{(1)}_{\widehat{p}-},\rho^{(2)}_{\widehat{p}+}$

$$\begin{aligned}\mathcal{D}_\theta\rho^{(2)}_{+\hat{p}} + i[\varphi^{\hat{p}}_{\hat{q}},\rho^{(2)}_{+\hat{q}}] &= 0\\ \mathcal{D}_\theta\rho^{(1)}_{-\hat{p}} + i[\varphi^{\hat{p}}_{\hat{q}},\rho^{(1)}_{-\hat{q}}] &= 0\,,\end{aligned} \tag{6.3.17}$$

which correspond to the supersymmetric counterparts to Nahm's equations (6.2.39). We will use these localizing equations below to expand the fermionic fields in terms of vectors in the tangent space to the moduli space of Nahm's equations $\mathcal{M}_k$.

Finally we drop the order r kinetic terms of the 4-dimensional gauge field and scalars $\varphi,\bar{\varphi}$ (which contribute only at order r), and we are left with the terms of order $\frac{1}{r}$ which describe the 4-dimensional action

$$\begin{aligned} S_{r^{-1}} = &-\frac{1}{4r\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\left(\mathcal{D}^{\mu'}\varphi^{\hat{a}}\mathcal{D}_{\mu'}\varphi_{\hat{a}} + \partial^\mu A_\theta \partial_\mu A_\theta\right) \\ &-\frac{2i}{r\ell}\int d\theta d^4x\sqrt{|g|}\,\mathrm{Tr}\left(\rho^{(1)}_{\hat{p}-}\gamma^\mu \mathcal{D}_\mu \rho^{(2)\hat{p}}\right). \end{aligned} \tag{6.3.18}$$

The remaining task is to express this action in terms of the fields $X = \{X^I\}$ and the massless fermionic degrees of freedom, and to integrate out the 4-dimensional components of the gauge field A_μ and the scalars $\varphi, \bar{\varphi}$, which appear as auxiliary fields in the 4-dimensional action. The subleading terms (at order r) in the $\varphi^{\widehat{a}}$ expansion can similarly be integrated out without producing any term in the final 4-dimensional action, so we ignore these contributions in the rest of the derivation.

6.3.3 Scalars

We will now describe the 4-dimensional theory in terms of 'collective coordinates', similar to the approach taken in e.g. ([171–173]) for dimensional reduction of 4-dimensional Super-Yang-Mills theories on a Riemann surface. The resulting theory is a (supersymmetric) sigma-model (6.3.1), where we will consider $M_4 = \mathbb{R}^4$. Let X^I be coordinates on the moduli space $\mathcal{M}$. The three scalar fields $\varphi_{\widehat{a}}$ and A_θ in terms of these collective coordinates are expanded as follows

$$\begin{aligned} \delta\varphi^{\widehat{a}} &= \Upsilon^{\hat{a}}_I \delta X^I \\ \delta A_\theta &= \Upsilon^\theta_I \delta X^I\,, \end{aligned} \tag{6.3.19}$$

where $I = 1, \ldots, 4k$, where we expanded in a basis of the cotangent bundle of $\mathcal{M}_k$, which up to gauge transformations is

$$\begin{aligned} \Upsilon^{\widehat{a}}_I &= \frac{\partial\varphi^{\widehat{a}}}{\partial X^I} + [E_I, \varphi^{\widehat{a}}] \\ \Upsilon^\theta_I &= \frac{\partial A_\theta}{\partial X^I} - \mathcal{D}_\theta E_I\,, \end{aligned} \tag{6.3.20}$$

where E_I defines a connection $\nabla_I = \partial_I + [E_I, \cdot]$ on $\mathcal{M}_k$. So we see that the expansion (6.3.19) is just the usual variation of fields on $\mathcal{M}_k$, e.g. $\delta\varphi^{\hat{a}} = \nabla_I \varphi^{\hat{a}} \delta X^I$. To guarantee, that the fields $\varphi^{\hat{a}}$ and A_θ satisfy the Nahm equations, the sections of the cotangent bundle to $\mathcal{M}_k$ have to solve

$$\mathcal{D}_\theta \Upsilon^{\widehat{a}}_I + \left[\Upsilon^\theta_I, \varphi^{\widehat{a}}\right] = \epsilon^{\hat{a}\hat{b}\hat{c}}\left[\Upsilon_{I\widehat{b}}, \varphi_{\widehat{c}}\right]\,, \tag{6.3.21}$$

The metric on $\mathcal{M}_k$ can be expressed in terms of these one-forms as

$$G_{IJ} = -\int d\theta\,\mathrm{Tr}(\Upsilon^{\widehat{a}}_I \Upsilon_{J\widehat{a}} + \Upsilon^\theta_I \Upsilon^\theta_J)\,. \tag{6.3.22}$$

Similarly we can write down an expression for the three symplectic forms (see e.g. ([168]))

$$\omega^{\widehat{a}}_{IJ} = \int d\theta \, \mathrm{Tr}(\epsilon^{\widehat{a}\widehat{b}\widehat{c}} \Upsilon_{I\widehat{b}} \Upsilon_{J\widehat{c}} + \Upsilon^{\widehat{a}}_I \Upsilon^{\theta}_J - \Upsilon^{\theta}_I \Upsilon^{\widehat{a}}_J) \, . \tag{6.3.23}$$

These provide the hyper-Kähler structure of the moduli space $\mathcal{M}$. In the appendix B.5 we collect various useful properties for these structures. Thanks to the gauge fixing condition

$$\mathcal{D}_\theta \Upsilon^\theta_I + \left[\varphi_{\widehat{a}}, \Upsilon^{\widehat{a}}_I\right] = 0 \, , \tag{6.3.24}$$

by substituting the expansions in collective coordinates (6.3.19) and (6.3.20) into the bosonic part of the action (6.3.18) we obtain

$$S_{scalars} = -\frac{1}{4r\ell} \int d^4x d\theta \sqrt{|g|} \, \mathrm{Tr} \left(\partial_I A_\theta \partial_J A_\theta + \partial_I \varphi^{\widehat{a}} \partial_J \varphi_{\widehat{a}} \right) \partial_\mu X^I \, \partial^\mu X^J \, . \tag{6.3.25}$$

The terms additional to the usual kinetic term vanish after integrating out the gauge field.

6.3.4 Fermions

The fermions satisfy the equation (6.3.17), which is the supersymmetry variation of the Nahm equations. The spinors therefore take values in the cotangent bundle to the moduli space $\mathcal{M}_k$ and we can therefore expand them in the basis that we defined in (6.3.20),

$$\begin{aligned} \rho^{(1)}_{-\hat{p}} &= \Upsilon^{\widehat{a}}_I (\sigma_{\widehat{a}})^{\widehat{q}}_{\widehat{p}} \lambda^{(1)I} + i \Upsilon^\theta_{I\widehat{p}} \lambda^{(1)I} \\ \rho^{(2)}_{+\hat{p}} &= \Upsilon^{\widehat{a}}_I (\sigma_{\widehat{a}})^{\widehat{q}}_{\widehat{p}} \lambda^{(2)i} + i \Upsilon^\theta_{I\widehat{p}} \lambda^{(2)I} \, , \end{aligned} \tag{6.3.26}$$

where $\lambda^{(1)I}, \lambda^{(2)I}$ are spacetime spinors, valued in $T\mathcal{M}_k$. The identities (B.5.2) imply that the fermionic fields obey the constraints

$$\omega^{\widehat{a}I}{}_J \lambda^{(i)J}_{\widehat{p}} = i(\sigma^{\widehat{a}})^{\widehat{q}}_{\widehat{p}} \lambda^{(i)I}_{\widehat{q}} \, . \tag{6.3.27}$$

The expansion in (6.3.26) can be seen to satisfy the equation of motion for the spinors (6.3.17) by making use of (6.3.21) and the gauge fixing condition (6.3.24). Then substituting the expansions in collective coordinates (6.3.26) in the fermionic part of the action (6.3.18) we find

$$\begin{aligned} S_{\rho_{kin}} = \frac{8i}{rl} \int d^4x \sqrt{|g|} \Big[& G_{IJ} \lambda^{(1)I\widehat{p}} \gamma^\mu \partial_\mu \lambda^{(2)J}_{\widehat{p}} \\ & - \int d\theta \mathrm{Tr} \left(\Upsilon^{\widehat{a}}_I \partial_J \Upsilon_{\widehat{a}K} + \Upsilon^\theta_I \partial_J \Upsilon^\theta_K \right) \lambda^{(1)I\widehat{p}} \gamma^\mu \lambda^{(2)K}_{\widehat{p}} \partial_\mu X^J \Big] \, . \end{aligned} \tag{6.3.28}$$

6.3.5 4-dimensional Sigma-Model Action and Symmetries

Finally, we need to integrate out the gauge field and the scalars φ, $\bar{\varphi}$, which is done in appendix B.5. The conclusion is that, in addition to giving the standard kinetic term for the scalars, this covariantizes the fermion action and results in a quartic fermion interaction that depends on the Riemann tensor of the moduli space. In summary we find the action

$$S_{4d} = \frac{1}{r\ell}\int_{M_4} d^4x\sqrt{|g_4|}\,\Big[\frac{1}{4}G_{IJ}\Big(\partial_\mu X^I\,\partial^\mu X^J + 8i\lambda^{(1)I\widehat{p}}\sigma^\mu \mathcal{D}_\mu \lambda^{(2)J}_{\widehat{p}}\Big) - 32R_{IJKL}\lambda^{(1)I\widehat{p}}\lambda^{(1)J}_{\widehat{p}}\lambda^{(2)K\widehat{q}}\lambda^{(2)L}_{\widehat{q}}\Big] \tag{6.3.29}$$

where $\mathcal{D}_\mu \lambda^{(2)I}_{\widehat{p}} = \partial_\mu \lambda^{(2)I}_{\widehat{p}} + \lambda^{(2)J}_{\widehat{p}}\Gamma^I_{JK}\partial_\mu X^K$. The final step is to decompose the spinors $\lambda^{(i)}$, as explained in appendix B.1.2, into 4-dimensional Weyl spinors

$$\lambda^{(1)I}_{\widehat{p}} = \frac{1}{4}\begin{pmatrix}\xi^{(1)}_{\widehat{p}}\\ 0\end{pmatrix}, \quad \lambda^{(2)I}_{\widehat{p}} = \frac{1}{4}\begin{pmatrix}0\\ \xi^{(2)}_{\widehat{p}}\end{pmatrix}, \tag{6.3.30}$$

obeying the reality conditions

$$(\xi^{(1)p})^* = \xi^{(2)}_{\dot{p}}\,, \quad (\xi^{(2)\dot{p}})^* = \xi^{(1)}_{p}\,. \tag{6.3.31}$$

The 4-dimensional sigma-model action from flat M_4 into the monopole moduli space $\mathcal{M}_k$ is then given by

$$S_{4d} = \frac{1}{4r\ell}\int_{M_4} d^4x\sqrt{|g_4|}\,\Big[G_{IJ}\Big(\partial_\mu X^I\,\partial^\mu X^J - 2i\xi^{(1)I\widehat{p}}\sigma^\mu\mathcal{D}_\mu\xi^{(2)J}_{\widehat{p}}\Big) - \frac{1}{2}R_{IJKL}\xi^{(1)I\widehat{p}}\xi^{(1)J}_{\widehat{p}}\xi^{(2)K\widehat{q}}\xi^{(2)L}_{\widehat{q}}\Big] \tag{6.3.32}$$

The supersymmetry transformations become

$$\begin{aligned} \delta X^I &= -i\left(\epsilon^{(2)\widehat{p}}\xi^{(1)I}_{\widehat{p}} + \epsilon^{(1)\widehat{p}}\xi^{(2)I}_{\widehat{p}}\right)\\ \delta\xi^{(1)I}_{\widehat{p}} &= \frac{1}{4}\left(\partial_\mu X^I\sigma^\mu\epsilon^{(1)}_{\widehat{p}} - i\omega^{\widehat{a}I}{}_J(\sigma_{\widehat{a}})^{\widehat{q}}_{\widehat{p}}\partial_\mu X^J\sigma^\mu\epsilon^{(1)}_{\widehat{q}}\right) - \Gamma^I_{jk}\delta X^J\xi^{(1)K}_{\widehat{p}}\\ \delta\xi^{(2)I}_{\widehat{p}} &= \frac{1}{4}\left(\partial_\mu X^I\bar{\sigma}^\mu\epsilon^{(2)}_{\widehat{p}} - i\omega^{\widehat{a}I}{}_J(\sigma_{\widehat{a}})^{\widehat{q}}_{\widehat{p}}\partial_\mu X^J\bar{\sigma}^\mu\epsilon^{(2)}_{\widehat{q}}\right) - \Gamma^I_{jk}\delta X^J\xi^{(2)K}_{\widehat{p}}. \end{aligned} \tag{6.3.33}$$

We have thus shown, that the M5-brane theory reduced on an S^2 gives rise to a 4-dimensional sigma-model with $\mathcal{N} = 2$ supersymmetry, based on maps from $\mathbb{R}^4$ into the moduli space $\mathcal{M}_k$ of Nahm's equations (with $\rho = [k]$ boundary conditions).

Chapter 7

Conclusion

In this thesis aspects of both F-Theory and M-Theory were investigated. The main aim was to gain insight into non-perturbative phenomena in string theory which can be accessible from these two frameworks.

In the F-Theory context, we saw how to properly take into account configuration of 7-branes in Type IIB compactifications. The low energy dynamics of the resulting compactification turns out to be encoded in the geometry of an elliptically fibered Calabi Yau. In particular, in order to produce a non-abelian gauge group in the resulting theory, the Calabi Yau manifold needs to develop geometric singularities. Up to subtleties in higher codimension in the base, we saw that an ADE classification determines both the singularity type in codimension one and the gauge group of the theory obtained upon compactification.

When considering compactifications to 4 dimensions, resulting in $\mathcal{N} = 1$ supersymmetry, we saw that phenomenological reasons led us to consider additional abelian factors to the ADE gauge group. Indeed, if the gauge group of the Standard Model can be embedded into $SU(5)$ or $SO(10)$ in Grand Unified Theories, it is also the case that additional gauge bosons are predicted which would give rise, for example, to unwanted proton decay operators. Such a problem can be solved by requiring the existence of additional abelian factors in the gauge group of the resulting theory, through which the proton decay operators result to be not gauge invariant.

The principle of geometric engineering is of use in this case as well. The existence of additional sections of the elliptic fibration results exactly in $U(1)$ factors through the following mechanism: via F-Theory/M-Theory duality the C_3 form of M-Theory can be decomposed along the forms which are Poincare dual to the sections of the fibration (which are divisors of the total variety), thus resulting in a gauge field for each additional rational section. A classical analysis through the application of Tate's algorithm allows to determine the singularities of the elliptic fibration. In this thesis, Tate's algorithm was applied to an elliptic fibration with two additional rational sections. The analysis resulted in a thorough classification of such elliptic fibrations, making use of the fact that the coefficients of the fibration belong to a unique factorization domain.

On the other hand, in the context of M-Theory, aspects connected to the theory on parallel membranes were investigated. M2 and M5-branes in M-Theory are half BPS solutions to 11-dimensional supergravity and their low energy dynamics is governed by, respectively, three and 6-dimensional superconformal field theories. Such theories have been elusive for many years, even though recently, following the breakthrough of the BLG model (followed by the ABJM one), the worldsheet theory on coincident M2-branes has been better understood.

The theory on parallel M5-branes, instead, still remains unknown, and aspects of it have been the focus of this thesis. A first line of research consisted in the extension of a previously proposed model ([6]), which similarly to what happens for the BLG model, realised the (2,0) supersymmetry of the 6-dimensional theory describing M5-branes through a gauge symmetry depending on a 3-algebra, rather than a usual Lie algebra. In this thesis, such realization of the (2,0) algebra was extended through the introduction of an abelian 3-form. After showing that such algebra closes on shell, it was seen that by solving the constraints for the fields that arise from closure, a natural dimensional reduction to 3 dimensions arises. Remarkably, upon solving such constraints, the (2,0) theory reduces to the BLG model. Therefore, the algebraic structure here proposed seems to include details of the dynamics of both parallel M2 and M5-branes.

The second line of research followed in the M-Theory context tried to extend a web of dualities which arises from the compactification of the theory on coincident M5-branes, the (2,0) theory. Indeed, even if a satisfying description of the (2,0) theory is not available, a number of results has been obtained by relating different compactifications to lower dimensions. In particular, through the breakthrough of the AGT correspondence, quantities of 4-dimensional $\mathcal{N} = 2$ theories were related to quantities in non-supersymmetric Toda theories in 2 dimensions. Dualities in other dimensions were then proposed following a similar reasoning. In this thesis, we begin the investigation of a new set of dualities, which would relate 2-dimensional theories obtained by compactifying the (2,0) theory on four-manifolds with the theory obtained by reducing the (2,0) theory on a two-sphere. Such dimensional reduction is here carried out to obtain a 4-dimensional sigma model into the moduli space of magnetic monopoles.

Appendix A

Appendices to Chapter 3

A.1 Solving Polynomial Equations over UFDs

In this appendix details are included of how to solve polynomial equations in the sections $\mathfrak{s}_i$ given that they belong to a unique factorization domain [73]. These solutions were repeatedly used in the algorithm to enhance the vanishing order of the discriminant. For convenience a part of this section will be a summary of the details given in the appendix A of [22], however there are polynomials specific to the case of two additional rational sections and the derivation of the solution for these is provided here. For more details on polynomial equations over UFDs that arise in the application of Tate's algorithm the reader is referred to appendix B of [21].

In [22] solutions were obtained for a three-term polynomial of the form

$$s_1^2 s_2 - s_1 s_3 s_4 + s_3^2 s_5 = 0\,. \tag{A.1.1}$$

Four solutions were found, three of which involve setting pairs of terms to zero, which are what we refer to as canonical solutions of the polynomials, and one other solution which we refer to as the non-canonical solution. The canonical solutions were found to be the pairs

$$\begin{aligned} s_1 &= s_3 = 0 \\ s_1 &= s_5 = 0 \\ s_2 &= s_3 = 0\,. \end{aligned} \tag{A.1.2}$$

The non-canonical solution is when

$$\begin{aligned} s_1 &= \sigma_1\sigma_2 \\ s_2 &= \sigma_3\sigma_4 \\ s_3 &= \sigma_1\sigma_3 \\ s_4 &= \sigma_2\sigma_4 + \sigma_3\sigma_5 \\ s_5 &= \sigma_2\sigma_5\,, \end{aligned} \tag{A.1.3}$$

where σ_2 and σ_3 are coprime over this UFD.

The non-canonical solution of a two-term polynomial was also needed

$$s_1 s_2 - s_3 s_4 = 0 : \quad \begin{cases} s_1 = \sigma_1\sigma_2 \\ s_2 = \sigma_3\sigma_4 \\ s_3 = \sigma_1\sigma_3 \\ s_4 = \sigma_2\sigma_4\,. \end{cases} \tag{A.1.4}$$

With this solution σ_2 and σ_3 are coprime, and so are σ_1 and σ_4.

A.1.1 Two Term Polynomial

We now look at the polynomial

$$P = s_1^2 - 4s_2s_3\,. \tag{A.1.5}$$

Setting $P = 0$ imposes the following conditions:

- There is an equality between the irreducible components of s_1^2 and the product of the irreducibles of s_2 and s_3.
- Write μ for the irreducible components common to all the three terms.
- Write σ_1 for the irreducible components common to s_1 and s_2.
- Write σ_2 for the irreducible components common to s_1 and s_3.

Note that no conclusion is drawn about irreducibles shared only by s_2 and s_3. Then the most general solution takes the form

$$s_1^2 - 4s_2s_3 = 0 : \quad \begin{cases} s_1 = 2\mu\sigma_1\sigma_2 \\ s_2 = \mu\sigma_1^2 \\ s_3 = \mu\sigma_2^2\,. \end{cases} \tag{A.1.6}$$

Since μ is the greatest common divisor of s_2 and s_3 we have that σ_1 and σ_2 are coprime.

A.1.2 Perfect Square Polynomial

The first perfect square polynomial is given by

$$s_1^2 - 4s_2s_3 = p^2 \,. \tag{A.1.7}$$

This can be reformulated as

$$(s_1 + p)\,(s_1 - p) = 4s_2s_3 \,, \tag{A.1.8}$$

which can be solved in general by applying the solution of the two-term polynomial (A.1.4). In this case, it reads

$$\begin{aligned} s_1 - p &= 2\sigma_1\sigma_2 \\ s_1 + p &= 2\sigma_3\sigma_4 \\ s_2 &= \sigma_1\sigma_3 \\ s_3 &= \sigma_2\sigma_4 \,. \end{aligned} \tag{A.1.9}$$

From the first two of these equations, one finds the generic form of s_1

$$s_1 = \sigma_1\sigma_2 + \sigma_3\sigma_4 \,. \tag{A.1.10}$$

So the general solution to the perfect square condition is

$$s_1^2 - 4s_2s_3 = p^2 : \quad \begin{cases} s_1 = \sigma_1\sigma_2 + \sigma_3\sigma_4 \\ s_2 = \sigma_1\sigma_3 \\ s_3 = \sigma_2\sigma_4 \,. \end{cases} \tag{A.1.11}$$

It follows from the solution of (A.1.4) that σ_2 and σ_3 are coprime, as are σ_1 and σ_4.

A.1.3 Three Term Polynomial

The three-term polynomial

$$P = s_1^2s_2s_3 - s_1s_4s_5 + s_5^2s_6 \,, \tag{A.1.12}$$

appears in the algorithm. By imposing $P = 0$ it is seen that $s_1 \mid s_5^2s_6$, since it divides the other two terms in the equation. In the same way $s_5 \mid s_1^2s_2s_3$. Decompose $s_5 = \sigma_1\sigma_2$ and

$s_1 = \sigma_1\sigma_3$, where $\sigma_1 = (s_1, s_5)$ is the greatest common divisor of the two terms, so that σ_2 and σ_3 have no common irreducibles. Then the equation of the polynomial becomes

$$\sigma_1^2(s_6\sigma_2^2 - s_4\sigma_2\sigma_3 + s_2s_3\sigma_3^2) = 0\,. \tag{A.1.13}$$

Applying the same reasoning it is now seen that $\sigma_3 \mid s_6\sigma_2$, but since σ_2 and σ_3 have no common irreducibles one can conclude that $\sigma_3 \mid s_6$. In the same way it can be deduced that $\sigma_2 \mid s_2s_3$. This can be expressed as

$$s_6 = \sigma_4\sigma_3, \qquad s_2s_3 = \kappa\sigma_2\,, \tag{A.1.14}$$

where κ is some constant of proportionality. The two-term solution (A.1.4) can be applied to the second of these equations to obtain

$$s_2 = \sigma_5\sigma_6, \quad s_3 = \sigma_7\sigma_8, \quad \kappa = \sigma_5\sigma_7, \quad \sigma_2 = \sigma_6\sigma_8\,. \tag{A.1.15}$$

Then the initial polynomial reduces to

$$\sigma_1^2\sigma_3\sigma_6\sigma_8(\sigma_3\sigma_5\sigma_7 + \sigma_4\sigma_6\sigma_8 - s_4) = 0\,, \tag{A.1.16}$$

from which can be solved for s_4. Then there is a non-canonical solution

$$s_1^2s_2s_3 - s_1s_4s_5 + s_5^2s_6 = 0: \quad \begin{cases} s_1 = \sigma_1\sigma_3 \\ s_2 = \sigma_5\sigma_6 \\ s_3 = \sigma_7\sigma_8 \\ s_4 = \sigma_3\sigma_5\sigma_7 + \sigma_4\sigma_6\sigma_8 \\ s_5 = \sigma_1\sigma_6\sigma_8 \\ s_6 = \sigma_3\sigma_4\,, \end{cases} \tag{A.1.17}$$

where the pairs (σ_5, σ_8), (σ_6, σ_7), and $(\sigma_3, \sigma_6\sigma_8)$ are all coprime. There are also four different canonical solutions

$$\begin{aligned} \sigma_1 = 0: &\quad s_1 = s_5 = 0 \\ \sigma_3 = 0: &\quad s_1 = s_6 = 0 \\ \sigma_6 = 0: &\quad s_2 = s_5 = 0 \\ \sigma_8 = 0: &\quad s_3 = s_5 = 0\,. \end{aligned} \tag{A.1.18}$$

A.2 Matter Loci of $SU(5)$ Models

In this appendix we list the matter loci of the I_5 fibers whose $U(1)$ charges are studied in section 3.4.

$$\sigma_3^2 s_{1,2} - \sigma_2\sigma_3 s_{2,2} + \sigma_2^2 s_{3,2} \tag{A.2.1}$$

$$\sigma_4^2 s_{1,2} - \sigma_4\sigma_5 s_{2,2} + \sigma_5^2 s_{3,2} \tag{A.2.2}$$

$$\sigma_1(\sigma_3^2 s_{5,1} - \sigma_2\sigma_3 s_{6,1} + \sigma_2^2 s_{7,1}) - (\sigma_2\sigma_4 - \sigma_3\sigma_5)(\sigma_2 s_{9,1} - \sigma_3 s_{8,1}) \tag{A.2.3}$$

$$\sigma_3^2\sigma_4(\sigma_4 s_{3,1} - \sigma_1 s_{7,0}) + \sigma_2^2(\sigma_4^2 s_{1,3} - \sigma_1\sigma_4 s_{5,2} + \sigma_1^2 s_{8,1}) - \sigma_2\sigma_3(\sigma_4^2 s_{2,2} - \sigma_1\sigma_4 s_{6,1} + \sigma_1^2 s_{9,0}) \tag{A.2.4}$$

$$\sigma_3^2\sigma_4(\sigma_4 s_{1,2} - \sigma_1 s_{5,1}) + \sigma_2^2\sigma_4(\sigma_4 s_{3,2} - \sigma_1 s_{7,1}) - \sigma_2\sigma_3(\sigma_4^2 s_{2,2} - \sigma_1\sigma_4 s_{6,1} + \sigma_1^2 s_{9,0}) \tag{A.2.5}$$

$$\sigma_1^2\sigma_2^2 s_{1,3} - \sigma_1\sigma_2 s_{2,2} s_{5,1} + s_{3,1} s_{5,1}^2 \tag{A.2.6}$$

$$-\sigma_1\sigma_3^2\sigma_4 s_{5,1} - \sigma_2\sigma_3(\sigma_4^2 s_{3,1} - \sigma_1\sigma_4 s_{6,1} + \sigma_1^2 s_{8,1}) + \sigma_1\sigma_2^2(-\sigma_4 s_{7,1} + \sigma_1 s_{9,1}) \tag{A.2.7}$$

$$\sigma_1^2\sigma_3(\sigma_3^2 s_{1,4} - \sigma_2\sigma_3 s_{2,3} + \sigma_2^2 s_{3,2}) + \sigma_1(\sigma_2\sigma_4 - \sigma_3\sigma_5)(\sigma_3^2 s_{5,2} - \sigma_2\sigma_3 s_{6,1} + \sigma_2^2 s_{7,0}) + \sigma_3(\sigma_2\sigma_4 - \sigma_3\sigma_5)^2 s_{8,0} \tag{A.2.8}$$

$$\sigma_1(\sigma_2^2 s_{3,1} - \sigma_2\sigma_3 s_{6,1} + \sigma_3^2 s_{8,1}) - (\sigma_2\sigma_4 - \sigma_3\sigma_5)(\sigma_2 s_{7,1} - \sigma_3 s_{9,1}) \tag{A.2.9}$$

$$(\sigma_2\sigma_4 - \sigma_3\sigma_5)^2 s_{1,4} + \sigma_2\sigma_5 s_{2,2}^2 - (\sigma_2\sigma_4 - \sigma_3\sigma_5) s_{2,2} s_{5,2} + \sigma_3\sigma_4 s_{5,2}^2 \tag{A.2.10}$$

$$(\sigma_2\sigma_4 - \sigma_3\sigma_5) s_{1,5} - \sigma_1(\sigma_2\sigma_4 - \sigma_3\sigma_5)(\sigma_2 s_{5,3} - \sigma_3 s_{2,3}) + \sigma_1^2(\sigma_3^2 s_{3,1} - \sigma_2\sigma_3 s_{6,1} + \sigma_2^2 s_{8,1}) \tag{A.2.11}$$

$$(\xi_3\sigma_2 - \xi_2\sigma_3)(\xi_2^2 s_{1,3} - \xi_2\xi_3 s_{2,3} + \xi_3^2 s_{3,3}) + (\xi_2\xi_4 - \xi_3\xi_5)(\xi_2^2 s_{5,1} - \xi_2\xi_3 s_{6,1} + \xi_3^2 s_{7,1}) \tag{A.2.12}$$

$$\xi_3\xi_4\sigma_1^2 - \sigma_1\sigma_4 s_{5,1} + \sigma_4^2 s_{1,2} \tag{A.2.13}$$

$$\xi_2^2(\sigma_1 s_{5,1} - \sigma_4 s_{1,2}) + \xi_2\xi_3(\sigma_4 s_{2,2} - \sigma_1 s_{6,1}) + \xi_3^2(\sigma_1 s_{7,1} - \sigma_4 s_{3,2}) \tag{A.2.14}$$

$$\xi_2^2\xi_4 s_{5,1} + \xi_2\xi_3(\sigma_4 s_{8,1} - \xi_4 s_{6,1}) + \xi_3^2(\xi_4 s_{7,1} - \sigma_4 s_{9,1}) \tag{A.2.15}$$

$$\xi_2(-\xi_2\sigma_2 s_{2,2} + \xi_2\sigma_3 s_{3,2} + \xi_3\sigma_2 s_{6,1} - \xi_3\sigma_3 s_{7,1}) - \xi_3^2\sigma_2 s_{9,0} \tag{A.2.16}$$

$$\xi_2^3(\sigma_3 s_{1,3} - \xi_4 s_{2,2}) + \xi_2^2\xi_3(\xi_5 s_{2,2} - \sigma_3 s_{5,2} + \xi_4 s_{6,1}) + \xi_3^3\xi_5 s_{9,0} - \xi_2\xi_3^2(\xi_5 s_{6,1} - \sigma_3 s_{8,1} + \xi_4 s_{9,0}) \tag{A.2.17}$$

$$\xi_1\xi_2^2\xi_4 s_{3,1} + \xi_2\xi_3(\xi_4^2\sigma_4 - \xi_1\xi_4 s_{6,1} + \xi_1^2\sigma_2 s_{7,1}) + \xi_1\xi_3^2(\xi_4 s_{8,1} - \xi_1\sigma_2 s_{9,1}) \tag{A.2.18}$$

$$(\xi_1\xi_2\sigma_3 - \sigma_2\sigma_4)^2 s_{3,1} + \xi_3(\xi_1\xi_2\sigma_3 - \sigma_2\sigma_4)(\xi_1\sigma_2 s_{2,2} - \xi_1\sigma_3 s_{6,1} + \xi_4\sigma_3\sigma_4) + \xi_1^2\xi_3^2(\sigma_2^2 s_{1,3} - \sigma_2\sigma_3 s_{5,2} + \sigma_3^2 s_{8,1}) \tag{A.2.19}$$

$$\begin{aligned}&\xi_4\xi_6^3\xi_8^3\sigma_4 - \xi_3\xi_6^2\xi_8^2(\xi_5\xi_7\sigma_4 - s_{1,2} + \xi_4 s_{6,1})\\ &\quad + \xi_3^2\xi_6\xi_8(\xi_4\xi_7\xi_8\sigma_3 - \xi_5\xi_6 s_{5,2} + \xi_5\xi_7 s_{6,1} + \xi_4\xi_5\xi_6 s_{9,1}) - \xi_3^2\xi_5(\xi_7^2\xi_8\sigma_3 - \xi_5\xi_6^2 s_{8,2} + \xi_5\xi_6\xi_7 s_{9,1})\end{aligned} \tag{A.2.20}$$

$$\xi_3^3\xi_4\xi_7\sigma_3^3 - \xi_5\xi_6^3\xi_8\sigma_3\sigma_4^2 + \xi_3^2\xi_6(\xi_5^2\xi_6^2 s_{3,2} + \sigma_3^2(s_{1,2} - \xi_4 s_{6,1}) - \xi_5\sigma_3(\xi_7\sigma_3\sigma_4 + \xi_6 s_{2,2} - \xi_4\xi_6 s_{7,1}))$$
$$+ \xi_3\xi_6^2\sigma_4(\xi_4\xi_8\sigma_3^2 + \xi_5(\sigma_3 s_{6,1} - \xi_5\xi_6 s_{7,1})) \tag{A.2.21}$$

$$\delta_1(\xi_2^2 s_{1,2} - \xi_2\xi_3 s_{2,2} + \xi_3^2 s_{3,2}) - \delta_4(\xi_2^2 s_{5,1} - \xi_2\xi_3 s_{6,1} + \xi_3^2 s_{7,1}) \tag{A.2.22}$$

$$\xi_2^2(\sigma_1 s_{5,1} - \delta_1\sigma_3 s_{8,1}) + \xi_3^2(\sigma_2 s_{7,1} - \delta_1\delta_2 s_{9,1}) + \xi_2\xi_3(-\sigma_1 s_{6,1} + \delta_1\delta_2 s_{8,1} + \delta_1\delta_3 s_{9,1}) \tag{A.2.23}$$

$$\delta_2\delta_4^2\sigma_1(\delta_2\xi_3 - \delta_3\xi_2) + \delta_1^2(\delta_2^2 s_{1,2} - \delta_2\delta_3 s_{2,2} + \delta_3^2 s_{3,2}) - \delta_1\delta_4(\delta_2^2 s_{5,1} - \delta_2\delta_3 s_{6,1} + \delta_3^2 s_{7,1}) \tag{A.2.24}$$

A.3 Resolution of Generic Singular Fibers

In section 3.1 a table (table 3.4) of canonical forms for many of the different fiber types as originally denoted by Kodaira was presented. In this section is is shown by explicitly constructing the resolution that each of the forms is the stated fiber. Given the set of resolutions and the canonical vanishing orders, the resolved geometry is uniquely determined and the form of the resolved geometry will not be written explicitly. For the Cartan divisors the equations are given after the resolution process and they will intersect according to the fiber type of the singularity under consideration.

A.3.1 $I_{2k+1}^{s(0|^n 1|^m 2)}$ $(n + m \leq k)$

The generic form for the singular $I_{2k+1}^{s(0|^n 1|^m 2)}$ is $(2k+1-(n+m), k-n, m, k+1-m, 0, 0, n, 0)$, provided that $(m + n \leq k)$. The resolution process involves several steps. First perform the following blow ups

$$(x, y, z; \zeta_1), \quad (x, y, \zeta_i; \zeta_{i+1}) \qquad 1 \leq i < \min\{k - n, k + 1 - m\}\,. \tag{A.3.1}$$

If $n \neq 0$ then the following small resolutions can be applied

$$(x, z; \xi_1), \quad (x, \xi_i; \xi_{i+1}) \qquad 1 \leq i < n\,. \tag{A.3.2}$$

Similarly if $m \neq 0$ the small resolutions,

$$(y, z; \delta_1), \quad (y, \delta_j; \delta_{j+1}) \qquad 1 \leq j < m\,, \tag{A.3.3}$$

are possible. If both $n \neq 0$ and $m \neq 0$ we need to use both sets of resolutions are applied. The next step depends on the sign of the quantity $m - n - 1$. We call ζ_{max} the last exceptional divisor introduced in the initial blow ups, and from now on the index will be used as max $= \min\{k - n, k + 1 - m\}$ If it is positive then the resolutions,

$$(y, \zeta_{max}; \chi_1), \quad (y, \chi_r; \chi_{r+1}) \qquad 1 \leq r < m - n - 1\,, \tag{A.3.4}$$

are used. Whereas if negative then the resolutions are

$$(x, \zeta_{max}; \Omega_1), \quad (x, \Omega_r; \Omega_{r+1}) \qquad 1 \leq r < -(m-n-1)\,. \tag{A.3.5}$$

If the term is exactly zero then we do neither set. Finally the process can be completed with the resolutions

$$(y, \zeta_s; \psi_s) \qquad 1 \leq s < \max\,. \tag{A.3.6}$$

The Cartan divisors are listed, assuming that $n-m-1>0$,

Exceptional Divisor	Fiber Equation
z	$l_1 l_2 w s_{6,0} + l_2 s_{9,0}\zeta_1\delta_1 + l_1 s_{7,0}\zeta_1\xi_1$
$\zeta_{i<max}$	$s_{6,0}x$
ζ_{max}	$xs_{6,0} + s_{5,k+1-m}\zeta_{max-1}$
$\delta_{j<m}$	$l_2 s_{6,0} + s_{7,0}\zeta_1$
δ_m	$l_2 y s_{6,0} + y s_{7,0}\zeta_1\psi_1 + s_{3,m}\zeta_1^m\delta_{m-1}\psi_1^{m-1}$
$\xi_{i<n}$	$l_1 s_{6,0} + s_{9,0}\zeta_1$
ξ_n	$l_1 x s_{6,0} + \zeta_1(xs_{9,0} + s_{8,0}\zeta_1^{n-1}\xi_{n-1})$
$\chi_{r<m-n-1}$	$xs_{6,0} + s_{5,k+1-m}\zeta_{max-1}$
χ_{m-n-1}	$y(xs_{6,0} + s_{5,k+1-m}\zeta_{max-1}\psi_{max-1})+$ $+\zeta_{\max-1}^{m-n}\psi_{max-1}^{m-n-1}(xs_{2,k-n} + s_{1,2k+1-n-m}\zeta_{max-1}\psi_{max-1})\chi_{m-n-2}$
$\psi_{s<max}$	$s_{6,0}y$

Then the ordered set $(z, \xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_{max}, \chi_1, \cdots, \chi_{m-n-1}, \psi_{max}, \cdots, \psi_1\delta_m, \cdots, \delta_1)$ gives an $I_{2k+1}^{s(0|^n1|^m2)}$ fiber, where the divisors are listed in the canonical ordering for the Dynkin diagram. One gets the analogous result when $n-m-1<0$.

A.3.2 $I_{2k+1}^{s(0|^n1|^m2)}$ ($k < n+m \leq \lfloor \frac{2}{3}(2k+1) \rfloor$)

The generic form for the singular fibers of type $I_{2k+1}^{s(0|^n1|^m2)}$ with section separation of the form $m+n \leq \lfloor \frac{2}{3}(2k+1) \rfloor$ is given by $(2k+1-(m+n), m, m, n, 0, 0, n, 0)$, where it is assumed that $m \geq n$. In order to resolve the geometry the following set of resolutions is used

$$
\begin{aligned}
(x,z;\xi_1), \qquad (x,\xi_i;\xi_{i+1}) \qquad & 1 \le i < n \\
(y,z;\delta_1), \qquad (y,\delta_j;\delta_{j+1}) \qquad & 1 \le j < m \\
(x,\delta_r;\chi_r) \qquad & 1 \le r \le m \\
(x,\chi_m;\psi_1), \qquad (x,\psi_s;\psi_{s+1}) \qquad & 1 \le s < 2k-2m-n\,.
\end{aligned}
\tag{A.3.7}
$$

Notice that the first three set of resolutions (together with z) produce $2m+n+1$ Cartan divisors. The fourth set of resolutions is only necessary if $2k-2m-n>0$. Then the Cartan divisors in the most general case are

Exceptional Divisor	Fiber Equation
z	$l_1 l_2 w s_{6,0} + l_2 s_{9,0}\delta_1 + l_1 s_{7,0}\xi_1$
δ_1	$l_2 s_{6,0} + s_{7,0}\xi_1\xi_2^2\cdots\xi_n^n\chi_1$
$\delta_{j<m}$	$l_2 s_{6,0} + s_{7,0}\chi_{j-1}\chi_j$
δ_m	$l_2(ys_{6,0} + s_{2,m}\delta_{m-1}\chi_{m-1}) + \chi_{m-1}(ys_{7,0} + s_{3,m}\delta_{m-1}\chi_{m-1})\chi_m$
$\xi_{i<n}$	$l_1 s_{6,0} + s_{9,0}\delta_1$
ξ_n	$l_1 x s_{6,0} + x s_{9,0}\delta_1\chi_1 + l_1 s_{5,n}\delta_1^n\xi_{n-1}\chi_1^{n-1} + s_{8,n}\delta_1^{n+1}\xi_{n-1}\chi_1^n$
$\chi_{r<m}$	$xs_{6,0}$
χ_m	$ys_{6,0} + s_{2,m}\chi_{m-1}$
$\psi_{s<2k-2m-n}$	$ys_{6,0} + s_{2,m}\chi_{m-1}$
$\psi_{2k-2m-n}$	$xys_{6,0} + xs_{2,m}\chi_{m-1} + s_{1,2k+1-m-n}\psi_{2k-2m-n-1}\chi_{m-1}^{n-1}$

The ordered set $(z,\xi_1,\cdots,\xi_n,\chi_1,\cdots,\chi_{m-1},\psi_{2k-2m-n},\cdots,\psi_1,\chi_m,\delta_m,\cdots,\delta_1)$ gives an $I_{2k+1}^{s(0|^n1|^m2)}$ singular fiber.

A.3.3 $I_{2k}^{s(0|^n1|^m2)}$ $(n+m \le k, \quad m<k)$

The generic form for the singular fiber of type $I_{2k}^{s(0|^n1|^m2)}$, where $m+n\le k$, is given by $(2k-(n+m), k-n, m, k-m, 0, 0, n, 0)$. The analysis follows closely that carried out for $I_{2k+1}^{s(0|^n1|^m2)}$ where more details can be found. In order to resolve the geometry perform the resolutions

$$
\begin{aligned}
(x,y,z;\zeta_1), \quad (x,y,\zeta_i;\zeta_{i+1}) \qquad & 1 \le i < \min\{k-n,k-m\} \\
(x,z;\xi_1), \qquad (x,\xi_i;\xi_{i+1}) \qquad & 1 \le i < n \\
(y,z;\delta_1), \qquad (y,\delta_j;\delta_{j+1}) \qquad & 1 \le j < m\,.
\end{aligned}
\tag{A.3.8}
$$

Then the sign of the quantity $m-n$ and then use the according set of small resolutions, where the index in ζ_{max} again means the last exceptional divisor introduced in the blow ups, that is, $\max = \min\{k-n, k-m\}$

$$\begin{aligned} &(y, \zeta_{max}; \chi_1), \quad (y, \chi_r; \chi_{r+1}) \qquad 1 \leq r < m-n \\ &(x, \zeta_{max}; \Omega_1), \quad (x, \Omega_r; \Omega_{r+1}) \qquad 1 \leq r < -(m-n)\,. \end{aligned} \tag{A.3.9}$$

Finally the resolution process is completed with

$$(y, \zeta_s; \psi_s) \qquad 1 \leq s < \max\,. \tag{A.3.10}$$

The Cartan divisors are, assuming $m-n>0$,

Exceptional Divisor	Fiber Equation
z	$l_1 l_2 w s_{6,0} + l_2 s_{9,0}\zeta_1\delta_1 + l_1 s_{7,0}\zeta_1\xi_1$
$\zeta_{i<max}$	$s_{6,0}x$
ζ_{max}	$x s_{6,0} + s_{5,k-m}\zeta_{max-1}$
$\delta_{j<m}$	$l_2 s_{6,0} + s_{7,0}\zeta_1$
δ_m	$l_2 y s_{6,0} + y s_{7,0}\zeta_1\psi_1 + s_{3,0}\zeta_1^m\delta_{m-1}\psi_1^{m-1}$
$\xi_{i<n}$	$l_1 s_{6,0} + s_{9,0}\zeta_1$
ξ_n	$l_1 x s_{6,0} + \zeta_1(x s_{9,0} + s_{8,n}\zeta_1^{n-1}\xi_{n-1})$
$\chi_{r<m-n}$	$x s_{6,0} + s_{5,k-m}\zeta_{max-1}$
χ_{m-n}	$y(x s_{6,0} + s_{5,k-m}\zeta_{max-1}\psi_{max-1})+$ $+\zeta_{\max-1}^{m-n+1}\psi_{max-1}^{m-n}(x s_{2,k-n} + s_{1,2k-m-n}\zeta_{max-1}\psi_{max-1})\chi_{m-n-1}$
$\psi_{s<max}$	$s_{6,0}y$

Then the ordered set $(z, \xi_1, \cdots, \xi_n, \zeta_1, \cdots, \zeta_{max}, \chi_1, \cdots, \chi_{m-n-1}, \psi_{max}, \cdots, \psi_1, \delta_m, \cdots, \delta_1)$ gives an $I_{2k+1}^{s(0|^n 1|^m 2)}$, and again analogously for $m-n<0$. Notice that if $m=k$ and $n=0$ the vanishing orders $(k,k,k,0,0,0,0,0)$ specify the singular fibers $I_{2k}^{ns(01|^n 2)}$ as listed in table 3.4. The k small resolutions that resolve the singularity are

$$(y, z; \delta_1), \qquad (y, \delta_j; \delta_{j+1}) \qquad 1 \leq j < k\,. \tag{A.3.11}$$

The resolved geometry has $k+1$ Cartan divisors, $k-1$ of which will split if $s_{6,0}^2 - 4 s_{5,0} s_{7,0}$ is a perfect, non-zero, square.

A.3.4 $I_{2k}^{s(0|^n 1|^m 2)}$ $(n+m \leq \lfloor \frac{4}{3}k \rfloor)$

The generic form for the singular fibers of type $I_{2k}^{s(0|^n 1|^m 2)}$ with section separation such that $m+n \leq \lfloor \frac{4}{3}k \rfloor$ is given by $(2k-(m+n), m, m, n, 0, 0, n, 0)$, where it is assumed that $m \geq n$. In order to resolve the geometry the following set of resolutions is used

$$\begin{aligned} (x, z; \xi_1), \qquad & (x, \xi_i; \xi_{i+1}) && 1 \leq i < n \\ (y, z; \delta_1), \qquad & (y, \delta_j; \delta_{j+1}) && 1 \leq j < m \\ & (x, \delta_r; \chi_r) && 1 \leq r \leq m \\ (x, \chi_m; \psi_1), \qquad & (x, \psi_s; \psi_{s+1}) && 1 \leq s < 2k-2m-n-1\,. \end{aligned} \tag{A.3.12}$$

Notice that the first three sets of resolutions produce $2m+n+1$ Cartan divisors. The fourth set of resolutions is then necessary if $2k-2m-n-1 \neq 0$. The Cartan divisors in the most general case are

Exceptional Divisor	Fiber Equation
z	$l_1 l_2 w s_{6,0} + l_2 s_{9,0} \delta_1 + l_1 s_{7,0} \xi_1$
δ_1	$l_2 s_{6,0} + s_{7,0} \xi_1 \xi_2^2 \cdots \xi_n^n \chi_1$
$\delta_{j<m}$	$l_2 s_{6,0} + s_{7,0} \chi_{j-1} \chi_j$
δ_m	$l_2(y s_{6,0} + s_{2,m} \delta_{m-1} \chi_{m-1}) + \chi_{m-1}(y s_{7,0} + s_{3,m} \delta_{m-1} \chi_{m-1}) \chi_m$
$\xi_{i<n}$	$l_1 s_{6,0} + s_{9,0} \delta_1$
ξ_n	$l_1 x s_{6,0} + x s_{9,0} \delta_1 \chi_1 + l_1 s_{5,n} \delta_1^n \xi_{n-1} \chi_1^{n-1} + s_{8,n} \delta_1^{n+1} \xi_{n-1} \chi_1^n$
$\chi_{r<m}$	$x s_{6,0}$
χ_m	$y s_{6,0} + s_{2,m} \chi_{m-1}$
$\psi_{s<2k-2m-n-1}$	$y s_{6,0} + s_{2,m} \chi_{m-1}$
$\psi_{2k-2m-n-1}$	$x y s_{6,0} + x s_{2,m} \chi_{m-1} + s_{1,2k-m-n} \psi_{2k-2m-n-2} \chi_{m-1}^{n-2}$

The ordered set $(z, \xi_1, \cdots, \xi_n, \chi_1, \cdots, \chi_{m-1}, \psi_{2k-2m-n-1}, \cdots, \psi_1, \chi_m, \delta_m, \cdots, \delta_1)$ gives an $I_{2k}^{s(0|^n 1|^m 2)}$ type singular fiber.

A.3.5 $I_{2k+1}^{ns(012)}$

The generic form for $I_{2k+1}^{ns(012)}$ is $(2k+1, k+1, 0, k+1, 0, 0, 0, 0)$. The geometry is singular at $x = y = z = 0$ and it can be resolved by performing a blow up $(x, y, z; \zeta_1)$. This process can be repeated k times, with the i^{th} resolution being $(x, y, \zeta_{i-1}; \zeta_i)$. The Cartan divisors are then

Exceptional Divisor	Fiber Equation
z	$l_1wx(l_1xs_{3,0}+l_2ys_{6,0})+l_2^2wy^2s_{8,0}+xy(l_1xs_{7,0}+l_2ys_{9,0})\zeta_1$
$\zeta_{i\leq k}$	$x^2s_{3,0}+xys_{6,0}+y^2s_{8,0}$

It is easily seen by considering the projective relations introduced by the resolutions the ordered set $(z,\zeta_1,\cdots,\zeta_k)$ of Cartan divisors intersects in an $I_{2k+1}^{ns(012)}$. Notice that if $s_{6,0}^2-4s_{3,0}s_{8,0}$ is a perfect square, each of the fiber components along $\{\zeta_i=0\}$ splits into two, thus giving the split version $I_{2k+1}^{s(012)}$.

A.3.6 $I_{2k}^{ns(012)}$

The generic form for $I_{2k}^{ns(012)}$ is $(2k,k,0,k,0,0,0,0)$. The singular geometry can be blown up k times with the i^{th} resolution being $(x,y,\zeta_{i-1};\zeta_i)$. The Cartan divisors are

Exceptional Divisor	Fiber Equation
z	$l_1wx(l_1xs_{3,0}+l_2ys_{6,0})+l_2^2wy^2s_{8,0}+xy(l_1xs_{7,0}+l_2ys_{9,0})\zeta_1$
$\zeta_{i<k}$	$x^2s_{3,0}+xys_{6,0}+y^2s_{8,0}$
ζ_k	$x^2s_{3,0}+xys_{6,0}+y^2s_{8,0}+\zeta_{k-1}xs_{2,k}+\zeta_{k-1}ys_{5,k}s_{1,2k}\zeta_{k-1}^2$

The ordered set of $(k+1)$ Cartan divisors $(z,\zeta_1,\cdots,\zeta_k)$ gives an $I_{2k}^{ns(012)}$. If, in addition, $s_{6,0}^2-4s_{3,0}s_{8,0}$ is a perfect square the $(k-1)$ Cartan divisors along ζ_i split into two, giving an $I_{2k}^{s(012)}$ fiber.

A.3.7 $I_{2k+1}^{*s(0|1||2)}$

The generic forms for the singular fibers of type $I_{2k+1}^{*s(0|1||2)}$ are characterized by the vanishing orders $(k+2,k+2,k+1,1,1,0,1,0)$. In order to resolve the geometry perform the resolutions

$$\begin{gathered}(x,y,z;\zeta_1),\quad (z,\zeta_1;\zeta_2),\quad (x,z;\zeta_3),\quad (y,z;\zeta_4),\quad (y,\zeta_2;\delta_1)\\ (y,\delta_i;\delta_{i+1})\qquad 1\leq i\leq 2k\,.\end{gathered}\tag{A.3.13}$$

The Cartan divisors are

Exceptional Divisor	Fiber Equation
z	$l_1 s_{7,0}\zeta_3 + l_2 s_{9,0}\zeta_4$
ζ_1	$y s_{5,1}$
ζ_2	$s_{5,1} z\zeta_4 + x\zeta_1(x s_{7,0}\zeta_3 + s_{9,0}\zeta_4\delta_1)$
ζ_3	$x s_{9,0} + z(l_1 s_{5,1} + s_{8,1}\zeta_2)$
ζ_4	$y s_{7,0}$
$\delta_{i\leq 2k}$	$s_{7,0}\zeta_1 + s_{5,1}\zeta_4$
δ_{2k+1}	$y(s_{7,0}\zeta_1 + s_{5,1}\zeta_4) + \zeta_1^k\zeta_4^k(s_{3,k+1}\zeta_1 + s_{1,k+2}\zeta_4)\delta_{2k}$

The ordered set of divisors $(z, \zeta_3, \zeta_2, \delta_1, \cdots, \delta_{2k+1}, \zeta_1, \zeta_4)$ specifies an $I_{2k+1}^{*s(0|1||2)}$ fiber in the canonical ordering.

A.3.8 $I_{2k}^{*s(0|1||2)}$

The generic forms for the singular fibers of type $I_{2k+1}^{*s(0|1||2)}$ are given by the vanishing orders $(k+2, k+1, k+1, 1, 1, 0, 1, 0)$. In order to resolve the geometry the following resolutions are used

$$\begin{gathered}(x, y, z; \zeta_1), \quad (z, \zeta_1; \zeta_2), \quad (x, z; \zeta_3), \quad (y, z; \zeta_4), \quad (y, \zeta_2; \delta_1) \\ (y, \delta_i; \delta_{i+1}) \qquad 1 \leq i \leq 2k-1\,.\end{gathered} \tag{A.3.14}$$

The Cartan divisors are listed, where, as always, all coordinates that are constrained to be non-zero by the projective relations have been scaled to one,

Exceptional Divisor	Fiber Equation
z	$l_1 s_{7,0}\zeta_3 + l_2 s_{9,0}\zeta_4$
ζ_1	$y s_{5,1}$
ζ_2	$s_{5,1} z\zeta_4 + x\zeta_1(x s_{7,0}\zeta_3 + s_{9,0}\zeta_4\delta_1)$
ζ_3	$x s_{9,0} + z(l_1 s_{5,1} + s_{8,1}\zeta_2)$
ζ_4	$y s_{7,0}$
$\delta_{i<2k}$	$s_{7,0}\zeta_1 + s_{5,1}\zeta_4$
δ_{2k}	$y s_{7,0}\zeta_1 + y s_{5,1}\zeta_4 + s_{2,k+1}\zeta_1^k\zeta_4^k\delta_{2k-1}$

Then the ordered set $(z, \zeta_3, \zeta_2, \delta_1, \cdots, \delta_{2k}, \zeta_1, \zeta_4)$ is an $I_{2k}^{*s(0|1||2)}$ fiber.

A.3.9 $I_{2k+1}^{*s(01||2)}$

The standard forms for the $I_{2k+1}^{*s(01||2)}$ type of singular fibers are given through the vanishing orders $(k+3, k+2, k+2, 1, 1, 0, 0, 0)$. In order to resolve the geometry use the resolutions

$$\begin{aligned} &(x, y, z; \zeta_1), \quad (y, z; \zeta_2), \quad (\zeta_1, \zeta_2; \zeta_3), \quad (y, \zeta_1; \zeta_4), \quad (y, \zeta_3; \delta_1) \\ &\qquad\qquad\qquad\quad (y, \delta_i; \delta_{i+1}) \qquad 1 \leq i \leq 2k\,. \end{aligned} \tag{A.3.15}$$

The Cartan divisors are

Exceptional Divisor	Fiber Equation
z	$l_1 x^2 s_{7,0} \zeta_1 + l_2 \zeta_2 (l_2 w s_{8,0} + x s_{9,0} \zeta_1 \zeta_3)$
ζ_1	$s_{5,1} z + s_{8,0} \zeta_4$
ζ_2	$y s_{7,0}$
ζ_3	$s_{5,1} z \zeta_2 + \zeta_4 (s_{7,0} \zeta_1 + s_{8,0} \zeta_2 \delta_1)$
ζ_4	$y s_{5,1}$
$\delta_{i \leq 2k}$	$s_{7,0} \zeta_4 + s_{5,1} \zeta_2$
δ_{2k+1}	$y s_{5,1} \zeta_2 + y s_{7,0} \zeta_4 + s_{2,k+2} \zeta_2^{k+1} \zeta_4^k \delta_{2k}$

Then the ordered set $(z, \zeta_1, \zeta_3, \delta_1, \cdots, \delta_{2k+1}, \zeta_2, \zeta_4)$ intersects in an $I_{2k+1}^{*s(01||2)}$ type fiber.

A.3.10 $I_{2k}^{*s(01||2)}$

The generic forms for singular fibers of type $I_{2k}^{*s(01||2)}$ are given by the vanishing orders $(k+2, k+2, k+1, 1, 1, 0, 0, 0)$. The geometry is non-singular after the resolutions

$$\begin{aligned} &(x, y, z; \zeta_1), \quad (y, z; \zeta_2), \quad (\zeta_1, \zeta_2; \zeta_3), \quad (y, \zeta_1; \zeta_4), \quad (y, \zeta_3; \delta_1) \\ &\qquad\qquad\qquad\quad (y, \delta_i; \delta_{i+1}) \qquad 1 \leq i \leq 2k-1\,. \end{aligned} \tag{A.3.16}$$

The Cartan divisors after these resolutions take the form

Exceptional Divisor	Fiber Equation
z	$l_1 x^2 s_{7,0}\zeta_1 + l_2\zeta_2(l_2 w s_{8,0} + x s_{9,0}\zeta_1\zeta_3)$
ζ_1	$s_{5,1}z + s_{8,0}\zeta_4$
ζ_2	$y s_{7,0}$
ζ_3	$s_{5,1}z\zeta_2 + \zeta_4(s_{7,0}\zeta_1 + s_{8,0}\zeta_2\delta_1)$
ζ_4	$y s_{5,1}$
$\delta_{i<2k}$	$s_{7,0}\zeta_4 + s_{5,1}\zeta_2$
δ_{2k}	$y(s_{5,1}\zeta_2 + s_{7,0}\zeta_4) + \zeta_2^k\zeta_4^{k-1}(s_{1,k+2}\zeta_2 + s_{3,k+1}\zeta_4)\delta_{2k-1}$

The set of divisors $(z, \zeta_1, \zeta_3, \delta_1, \cdots, \delta_{2k}, \zeta_2, \zeta_4)$ then has the intersection structure of an $I_{2k}^{*s(01\|2)}$ fiber.

A.3.11 $I_{2k+1}^{*ns(01|2)}$

The generic forms for the singular fibers of type $I_{2k+1}^{*ns(01|2)}$ are given by the vanishing orders $(2k+3, k+2, 1, k+2, 1, 0, 0, 0)$. In order to resolve the geometry perform the resolutions

$$\begin{aligned} (x, y, z; \zeta_1), \quad (x, y, \zeta_i; \zeta_{i+1}) \qquad & 1 \le i \le k \\ (y, z; \delta_1), \qquad (y, \zeta_i; \delta_{i+1}) \qquad & 1 \le i \le k+1 \\ (\zeta_j, \delta_j; \xi_j) \qquad & 1 \le j \le k+1 \\ (\zeta_{k+1}, \delta_{k+2}; \chi)\,. \qquad & \end{aligned} \tag{A.3.17}$$

The Cartan divisors are

Exceptional Divisor	Fiber Equation
z	$l_1 x^2 s_{7,0}\zeta_1 + l_2\delta_1(l_2 w s_{8,0} + x s_{9,0}\zeta_1\xi_1)$
$\zeta_{i\le k}$	δ_{i+1}
δ_1	$s_{3,1}z + y s_{7,0}\delta_2$
δ_{k+2}	$x^2 s_{3,1} + x s_{2,k+2}\delta_{k+1}\xi_{k+1} + s_{1,2k+3}\delta_{k+1}^2\xi_{k+1}^2$
ξ_1	$s_{3,1}z\zeta_1 + (s_{7,0}\zeta_1 + s_{8,0}\delta_1)\delta_2$
$\xi_{j\le k+1}$	$s_{3,1}\zeta_{j-1}\zeta_j + s_{8,0}\delta_j\delta_{j+1}$
χ	$s_{8,0}\delta_{k+2} + \zeta_{k+1}(x^2 s_{3,1} + x s_{2,k+2}\xi_{k+1} + s_{1,2k+3}\xi_{k+1}^2)$

Then the set $(z, \delta_1, \xi_1, \zeta_1, \xi_2, \zeta_2, \cdots, \zeta_k, \xi_{k+1}, \chi, \delta_{k+2})$ is an $I_{2k+1}^{*ns(01|2)}$ fiber. Notice that if $s_{2,k+2}^2 - 4s_{1,2k+3}s_{3,1}$ is a perfect, non zero square then the Cartan divisor δ_{k+2} splits into two and the fiber is an $I_{2k+1}^{*s(01|2)}$.

A.3.12 $I_{2k}^{*ns(01|2)}$

The standard forms for the singular fibers of type $I_{2k}^{*ns(01|2)}$ are expressed through the vanishing orders $(2k+2, k+2, 1, k+1, 1, 0, 0, 0)$. The space is resolved by the following sequence of resolutions

$$\begin{aligned} (x,y,z;\zeta_1), \quad (x,y,\zeta_i;\zeta_{i+1}) \qquad & 1 \leq i \leq k \\ (y,z;\delta_1), \qquad (y,\zeta_i;\delta_{i+1}) \qquad & 1 \leq i \leq k \\ (\zeta_j,\delta_j;\xi_j) \qquad & 1 \leq j \leq k+1\,. \end{aligned} \tag{A.3.18}$$

The Cartan divisors in the resolved geometry are then

Exceptional Divisor	Fiber Equation
z	$l_1 x^2 s_{7,0}\zeta_1 + l_2\delta_1(l_2 w s_{8,0} + x s_{9,0}\zeta_1\xi_1)$
$\zeta_{i\leq k}$	δ_{i+1}
ζ_{k+1}	$y^2 s_{8,0} + y s_{5,k+1}\zeta_k + s_{1,2k+2}\zeta_k^2$
δ_1	$s_{3,1}z + y s_{7,0}\delta_2$
ξ_1	$s_{3,1}z\zeta_1 + \delta_2(s_{8,0}\delta_1 + s_{7,0}\zeta_1)$
$\xi_{j\leq k}$	$s_{3,1}\zeta_{j-1}\zeta_j + s_{8,0}\delta_j\delta_{j+1}$
ξ_{k+1}	$s_{3,1}\zeta_k\zeta_{k+1} + (y^2 s_{8,0} + y s_{5,k+1}\zeta_k + s_{1,2k+2}\zeta_k^2)\delta_{k+1}$

The ordered set $(z,\delta_1,\xi_1,\zeta_1,\xi_2,\zeta_2,\cdots,\zeta_k,\xi_{k+1},\zeta_{k+1})$ represents an $I_{2k}^{*ns(01|2)}$ fiber. We note that if $s_{5,k+2}^2 - 4s_{1,2k+2}s_{8,0}$ is a perfect, non-zero square then the Cartan divisor ζ_{k+1} splits into two and the fiber is an $I_{2k}^{*s(01|2)}$ fiber.

A.4 Determination of the Cubic Equation

In this appendix a non-singular elliptic curve with three marked points is constructed following [15, 174] and it is embedded into the projective space $\mathbb{P}^2$. This non-singular elliptic curve is then fibered over some arbitrary base, B_3, to create a non-singular elliptic fibration.

Begin by considering a genus one algebraic curve, X, with three marked divisors P, Q, and R. The line bundle $\mathcal{O}(P+Q+R)$ is identified with the vector space of meromorphic functions on X, with poles of at worst order one at the points P, Q, and R, and regular elsewhere. The Riemann-Roch theorem for algebraic curves fixes the dimension of such vector spaces. Any divisor in an algebraic curve X can be written as a formal sum over the points of X: $D = \sum_{P \in X} n_P P$, where $n_P = 0$ for all by finitely many P. The Riemann-Roch theorem then states that for any such divisor

$$\dim \mathcal{O}(D) = \deg(D) + 1 - g \,, \tag{A.4.1}$$

where $\deg(D)$ is the sum over the n_P associated to D. Thus it follows that the vector space $\mathcal{O}(P+Q+R)$ has dimension 3. Let the three generators of this space be denoted by the functions 1, x, and y. We can determine the pole structure of these functions. Consider first the vector space $\mathcal{O}(P)$, which has dimension 1 for any $P \in X$, and which must contain the 1-dimensional space of constant functions. As it has dimension 1 it can only contain these holomorphic functions, and therefore there are no functions with a pole of order one at any single point of X. The pole structure of 1, x, and y can then be determined to be as given in table A.4, up to linear combinations.

Function	Order		
	P	Q	R
1	0	0	0
x	1	1	0
y	1	0	1
xy	2	1	1
x^2	2	2	0
y^2	2	0	2
x^2y	3	2	1
xy^2	3	1	2
x^3	3	3	0
y^3	3	0	3

Similarly one can consider the vector space $\mathcal{O}(2(P+Q+R))$ which has degree, and thus dimension, 6. Clearly 1, x, and y are generators of half this space, and the other three generators can be written as x^2, y^2 and xy, which have the pole structures given in table A.4. Finally consider $\mathcal{O}(3(P+Q+R))$ which has dimension nine. Out of the six generators for $\mathcal{O}(2(P+Q+R))$ one can construct ten meromorphic functions inside $\mathcal{O}(3(P+Q+R))$, which must be linearly dependent for the space to be of dimension nine. We write this relation as

$$\begin{aligned} A_1 + A_2 x + A_3 y + &A_4 xy + A_5 x^2 + A_6 y^2 + A_7 x^2 y \\ &+ A_8 xy^2 + A_9 x^3 + A_{10} y^3 = 0 \,. \end{aligned} \tag{A.4.2}$$

The right-hand side of this equation is the zero function, which does not have poles anywhere. It must then be the case that the left-hand side must not have any poles for such a relation to hold. There are two terms with poles of order three at the points Q, R, which are the x^3 and y^3 terms respectively. There is no other term which contributes a pole of these orders and so could be tuned to cancel it off, therefore the only solution is to set the coefficients, A_9 and A_{10}, to zero.

This leaves exactly two terms with a pole of order three at P and, by the same argument as above, if either of these coefficients vanish then the other must also vanish. Let us follow this line of argument and demonstrate that it leads to a contradiction. If $A_7 = A_8 = 0$ then it is clear that both $A_5 = 0$ and $A_6 = 0$ as these are the only terms remaining with a pole of order two in Q, R. Further if these terms are vanishing the arguments above lead us to conclude that $A_4 = A_3 = A_2 = A_1 = 0$. If this is the case then this is not a non-trivial relation among these ten meromorphic functions, and so the relation cannot have either of A_7 or A_8 vanishing.

After the embedding of the elliptic curve into projective space the relation defines the curve by a hypersurface equation which we write as

$$\mathfrak{s}_1 w^3 + \mathfrak{s}_2 w^2 x + \mathfrak{s}_3 w x^2 + \mathfrak{s}_5 w^2 y + \mathfrak{s}_6 wxy + \mathfrak{s}_7 x^2 y + \mathfrak{s}_8 w y^2 + \mathfrak{s}_9 x y^2 = 0 \,, \tag{A.4.3}$$

where $[x : y : w]$ are the coordinates of a $\mathbb{P}^2$ and $\mathfrak{s}_i$ lie in some base coordinate ring R. This will be taken as the defining equation of our elliptic fibration.

The cubic equation (A.4.3) can always be mapped into the form of a Weierstrass model using Nagell's algorithm [175, 176]. For the convenience of the reader we write here only the f and g of the corresponding Weierstrass model. The complete derivation of the Weierstrass model from the cubic (A.4.3) is given in [1,24–26] and we do not repeat it here. The Weierstrass equation is

$$\widetilde{y}^2 = \widetilde{x}^3 + f\widetilde{x} + g \,, \tag{A.4.4}$$

where f and g are given in terms of the coefficients of (3.1.1) as

$$\begin{aligned} f =& \frac{1}{48}(-\mathfrak{s}_6^4 + 8\mathfrak{s}_6^2(\mathfrak{s}_5\mathfrak{s}_7 + \mathfrak{s}_3\mathfrak{s}_8 + \mathfrak{s}_2\mathfrak{s}_0) - 24\mathfrak{s}_6(\mathfrak{s}_2\mathfrak{s}_7\mathfrak{s}_8 + \mathfrak{s}_3\mathfrak{s}_5\mathfrak{s}_9 + \mathfrak{s}_1\mathfrak{s}_7\mathfrak{s}_9) \\ &+ 16(-\mathfrak{s}_5^2\mathfrak{s}_7^2 + 3\mathfrak{s}_1\mathfrak{s}_7^2\mathfrak{s}_8 - \mathfrak{s}_3^2\mathfrak{s}_8^2 + \mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_8\mathfrak{s}_9 - \mathfrak{s}_2^2\mathfrak{s}_9^2 + 3\mathfrak{s}_1\mathfrak{s}_3\mathfrak{s}_9^2 + \mathfrak{s}_5\mathfrak{s}_7(\mathfrak{s}_3\mathfrak{s}_8 + \mathfrak{s}_2\mathfrak{s}_9))) \end{aligned} \tag{A.4.5}$$

$$\begin{aligned} g =& \frac{1}{864}(\mathfrak{s}_6^6 - 12\mathfrak{s}_6^4(\mathfrak{s}_5\mathfrak{s}_7 + \mathfrak{s}_3\mathfrak{s}_8 + \mathfrak{s}_2\mathfrak{s}_9) + 36\mathfrak{s}_6^3(\mathfrak{s}_2\mathfrak{s}_7\mathfrak{s}_8 + \mathfrak{s}_3\mathfrak{s}_5\mathfrak{s}_9 + \mathfrak{s}_1\mathfrak{s}_7\mathfrak{s}_9) \\ &+ 24\mathfrak{s}_6^2(2\mathfrak{s}_5^2\mathfrak{s}_7^2 + 2\mathfrak{s}_3^2\mathfrak{s}_8^2 + \mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_8\mathfrak{s}_9 + 2\mathfrak{s}_2^2\mathfrak{s}_9^2 + \mathfrak{s}_5\mathfrak{s}_7(\mathfrak{s}_3\mathfrak{s}_8 + \mathfrak{s}_2\mathfrak{s}_9) - 3\mathfrak{s}_1(\mathfrak{s}_7^2\mathfrak{s}_8 + \mathfrak{s}_3\mathfrak{s}_9^2)) \\ &+ 8(-8\mathfrak{s}_5^3\mathfrak{s}_7^3 - 72\mathfrak{s}_1\mathfrak{s}_3\mathfrak{s}_7^2\mathfrak{s}_8^2 - 8\mathfrak{s}_3^3\mathfrak{s}_8^3 + 27\mathfrak{s}_1^2\mathfrak{s}_7^2\mathfrak{s}_9^2 - 72\mathfrak{s}_1\mathfrak{s}_3^2\mathfrak{s}_8\mathfrak{s}_9^2 - 8\mathfrak{s}_2^3\mathfrak{s}_9^3 \\ &+ 3\mathfrak{s}_2^2\mathfrak{s}_8(9\mathfrak{s}_7^2\mathfrak{s}_8 + 4\mathfrak{s}_3\mathfrak{s}_9^2) + 6\mathfrak{s}_5\mathfrak{s}_7(6\mathfrak{s}_1\mathfrak{s}_7^2\mathfrak{s}_8 + 2\mathfrak{s}_3^2\mathfrak{s}_8^2 + \mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_8\mathfrak{s}_9 + 2\mathfrak{s}_2^2\mathfrak{s}_9^2 - 3\mathfrak{s}_1\mathfrak{s}_3\mathfrak{s}_9^2) \\ &+ 6\mathfrak{s}_2\mathfrak{s}_9(-3\mathfrak{s}_1\mathfrak{s}_7^2\mathfrak{s}_8 + 2\mathfrak{s}_3^2\mathfrak{s}_8^2 + 6\mathfrak{s}_1\mathfrak{s}_3\mathfrak{s}_9^2) + 3\mathfrak{s}_5^2(4\mathfrak{s}_3\mathfrak{s}_7^2\mathfrak{s}_8 + 4\mathfrak{s}_2\mathfrak{s}_7^2\mathfrak{s}_9 + 9\mathfrak{s}_3^2\mathfrak{s}_9^2)) \\ &- 144\mathfrak{s}_6(\mathfrak{s}_2^2\mathfrak{s}_7\mathfrak{s}_8\mathfrak{s}_9 + \mathfrak{s}_9(\mathfrak{s}_1\mathfrak{s}_5\mathfrak{s}_7^2 + \mathfrak{s}_3^2\mathfrak{s}_5\mathfrak{s}_8 + \mathfrak{s}_3\mathfrak{s}_8(\mathfrak{s}_5^2 - 5\mathfrak{s}_1\mathfrak{s}_8)) \\ &+ \mathfrak{s}_2(\mathfrak{s}_5\mathfrak{s}_7^2\mathfrak{s}_8 + \mathfrak{s}_3\mathfrak{s}_7\mathfrak{s}_8^2 + \mathfrak{s}_1\mathfrak{s}_7\mathfrak{s}_9^2))) \,. \end{aligned} \tag{A.4.6}$$

Appendix B

Appendices to Chapter 6

B.1 Conventions and Spinor Decompositions

B.1.1 Indices

Our index conventions, for Lorentz and R-symmetry representations, which are used throughout Chapter 6 are summarized in the following tables. Note that R-symmetry indices are always hatted. Note that $\underline{m} = 1, \cdots, 8$, however only four components are independent for Weyl spinors in 6d.

Lorentz indices	6d	5d	4d	3d	2d
Curved vector	$\underline{\mu}, \underline{\nu}$	μ', ν'	μ, ν	.	.
Flat vector	$\underline{A}, \underline{B}$	A', B'	A, B	a, b	x, y
Spinors	$\underline{m}, \underline{n}$	m', n'	$p, q; \dot{p}, \dot{q}$	.	.
	($\mathbf{4}$ of $\mathfrak{su}(4)_L$)	($\mathbf{4}$ of $\mathfrak{sp}(4)_L$)	($\mathbf{2}$ of $\mathfrak{su}(2)_L$; $\mathfrak{su}(2)_{L'}$)		

Table B.1: Spacetime indices in various dimensions.

	$\mathfrak{so}(5)_R$	$\mathfrak{sp}(4)_R$	$\mathfrak{so}(3)_R$	$\mathfrak{su}(2)_R$	$\mathfrak{so}(2)_R \simeq \mathfrak{u}(1)_R$
Index for fundamental	$\widehat{A}, \widehat{B}$	$\widehat{m}, \widehat{n}$	$\widehat{a}, \widehat{b}$	$\widehat{p}, \widehat{q}$	$\widehat{x}, \widehat{y}$

Table B.2: R-symmetry indices.

B.1.2 Gamma-matrices and spinors: 6d, 5d and 4d

We work with the mostly + signature $(-,+,\cdots,+)$. The gamma matrices $\Gamma^{\underline{A}}$ in 6 dimensions, $\gamma^{A'}$ in 5 dimensions and γ^{A} in 4 dimensions, respectively, are defined as follows:

$$\begin{aligned}
\Gamma_1 = &\ i\sigma_2 \otimes \mathbf{1}_2 \otimes \sigma_1 &\equiv&\ \gamma_1 \otimes \sigma_1 \\
\Gamma_2 = &\ \sigma_1 \otimes \sigma_1 \otimes \sigma_1 &\equiv&\ \gamma_2 \otimes \sigma_1 \\
\Gamma_3 = &\ \sigma_1 \otimes \sigma_2 \otimes \sigma_1 &\equiv&\ \gamma_3 \otimes \sigma_1 \\
\Gamma_4 = &\ \sigma_1 \otimes \sigma_3 \otimes \sigma_1 &\equiv&\ \gamma_4 \otimes \sigma_1 \\
\Gamma_5 = &\ -\sigma_3 \otimes \mathbf{1}_2 \otimes \sigma_1 &\equiv&\ \gamma_5 \otimes \sigma_1 \\
\Gamma_6 = &\ \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_2 \,, &&
\end{aligned} \tag{B.1.1}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.1.2}$$

The 6-dimensional gamma matrices satisfy the Clifford algebra

$$\{\Gamma_{\underline{A}}, \Gamma_{\underline{B}}\} = 2\eta_{\underline{AB}} \,, \tag{B.1.3}$$

and similarly for the 5-dimensional and 4-dimensional gamma matrices.

Futhermore we define

$$\Gamma^{\underline{A}_1\underline{A}_2\cdots\underline{A}_n} \equiv \Gamma^{[\underline{A}_1\underline{A}_2\cdots\underline{A}_n]} = \frac{1}{n!}\sum_{w\in\mathcal{S}_n}(-1)^w \Gamma^{\underline{A}_{w(1)}}\,\Gamma^{\underline{A}_{w(2)}}\ldots\Gamma^{\underline{A}_{w(n)}} \,, \tag{B.1.4}$$

and similarly for all types of gamma matrices.

The chirality matrix in 4 dimensions is $\gamma_5 = -\sigma_3 \otimes \mathbf{1}_2$ and in 6 dimensions is defined by

$$\Gamma_7 = \Gamma^1\Gamma^2\cdots\Gamma^6 \;=\; \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_3 \,. \tag{B.1.5}$$

The charge conjugation matrices in 6 dimensions, 5 dimensions and 4 dimensions are defined by

$$\begin{aligned}
C_{(6\mathrm{d})} &= \sigma_3 \otimes \sigma_2 \otimes \sigma_2 &&\equiv \underline{C} \\
C_{(5\mathrm{d})} &= C_{(4\mathrm{d})} \;=\; -i\,\sigma_3 \otimes \sigma_2 &&\equiv C \,.
\end{aligned} \tag{B.1.6}$$

They obey the identities

$$\begin{aligned}
\left(\Gamma^{\underline{A}}\right)^T &= -\underline{C}\Gamma^{\underline{A}}\underline{C}^{-1}\,, \quad \underline{A} = 1, \cdots, 6. \\
\left(\gamma^{A'}\right)^T &= C\gamma^{A'}C^{-1}\,, \quad A' = 1, \cdots, 5. \\
\left(\gamma^{A}\right)^T &= C\gamma^{A}C^{-1}\,, \quad A = 1, \cdots, 4.
\end{aligned} \tag{B.1.7}$$

To define irreducible spinors we also introduce the B-matrices

$$\begin{aligned}
B_{(6\mathrm{d})} &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \\
B_{(5\mathrm{d})} &= B_{(4\mathrm{d})} = i\,\sigma_1 \otimes \sigma_2\,,
\end{aligned} \tag{B.1.8}$$

which satisfy

$$\begin{aligned}
\left(\Gamma^{\underline{A}}\right)^* &= B_{(6\mathrm{d})}\Gamma^{\underline{A}}B^{-1}_{(6\mathrm{d})}\,, \quad \underline{A} = 1, \cdots, 6. \\
\left(\gamma^{A'}\right)^* &= -B_{(5\mathrm{d})}\gamma^{A'}B^{-1}_{(5\mathrm{d})}\,, \quad A' = 1, \cdots, 5. \\
\left(\gamma^{A}\right)^* &= -B_{(4\mathrm{d})}\gamma^{A}B^{-1}_{(4\mathrm{d})}\,, \quad A = 1, \cdots, 4.
\end{aligned} \tag{B.1.9}$$

The 6-dimensional Dirac spinors have eight complex components. Irreducible spinors have a definite chirality and have only four complex components. For instance a spinor ρ of positive chirality satisfies $\Gamma^7\rho = \rho$. Similarly Dirac spinors in 4 dimensions have four complex components and Weyl spinors obey a chirality projection, for instance $\gamma_5\psi = \psi$ for positive chirality, and have two complex components. The components of positive, resp. negative, chirality spinors in 4 dimensions are denoted with the index $p = 1, 2$, resp. $\dot{p} = 1, 2$.

The indices of Weyl spinors in 6 dimensions can be raised and lowered using the SW/NE (South-West/North-Est) convention:

$$\rho^{\underline{m}} = \rho_{\underline{n}}\underline{C}^{\underline{nm}}\,, \quad \rho_{\underline{m}} = \underline{C}_{\underline{mn}}\rho^{\underline{n}}\,, \tag{B.1.10}$$

with $(\underline{C}^{\underline{mn}}) = (\underline{C}_{\underline{mn}}) = \underline{C}$. There is a slight abuse of notation here: the indices $\underline{m}, \underline{n}$ go from 1 to 8 here (instead of 1 to 4), but half of the spinor components are zero due to the chirality condition. We indices are omitted the contraction is implicitly SW/NE. For instance

$$\rho\tilde{\rho} = \rho_{\underline{m}}\tilde{\rho}^{\underline{m}}\,, \quad \rho\Gamma^{\underline{A}}\tilde{\rho} = \rho_{\underline{n}}(\Gamma^{\underline{A}})^{\underline{n}}{}_{\underline{m}}\tilde{\rho}^{\underline{m}}\,, \tag{B.1.11}$$

with $(\Gamma^{\underline{A}})^{\underline{n}}{}_{\underline{m}}$ the components of $\Gamma^{\underline{A}}$ as given above.

The conventions on 5-dimensional and 4-dimensional spinors are analogous: indices are raised and lowered using the SW/NE convention with $(C^{m'n'}) = (C_{m'n'}) = C$ in 5 dimensions and with the epsilon matrices $\epsilon^{pq} = \epsilon_{pq} = \epsilon^{\dot{p}\dot{q}} = \epsilon_{\dot{p}\dot{q}}$, with $\epsilon^{12} = 1$. They are contracted contracted using the SW/NE convention.

We also introduce Gamma matrices $\Gamma^{\widehat{A}}$ for the $\mathfrak{sp}(4)_R = \mathfrak{so}(5)_R$ R-symmetry

$$\Gamma^1 = \sigma_1 \otimes \sigma_3 \quad , \; \Gamma^2 = \sigma_2 \otimes \sigma_3 \quad , \; \Gamma^3 = \sigma_3 \otimes \sigma_3 \quad , \; \Gamma^4 = \mathbf{1}_2 \otimes \sigma_2 \quad , \; \Gamma^5 = \mathbf{1}_2 \otimes \sigma_1 \quad . \tag{B.1.12}$$

For the R-symmetry indices we use the opposite convention compare to the Lorentz indices, namely indices are raise and lowered with the NW/SE convention:

$$\rho_{\widehat{m}} = \rho^{\widehat{n}} \Omega_{\widehat{n}\widehat{m}} \, , \quad \rho^{\widehat{m}} = \Omega^{\widehat{m}\widehat{n}} \rho_{\widehat{n}} \, , \tag{B.1.13}$$

with $(\Omega_{\widehat{m}\widehat{n}}) = (\Omega^{\widehat{m}\widehat{n}}) \; = \; i\sigma_2 \otimes \sigma_1$. When unspecified, R-symmetry indices are contracted with the NW/SE convention, so that we have for instance $\rho\tilde{\rho} = \rho^{\widehat{m}}_m \tilde{\rho}^m_{\widehat{m}}$.

A collection of Weyl spinors $\rho_{\widehat{m}}$ in 6 dimensions transforming in the $\mathbf{4}$ of $\mathfrak{sp}(4)_R$ can further satisfy a Symplectic-Majorana condition (which exist in Lorentzian signature, but not in Euclidean signature)

$$(\rho_{\widehat{m}})^* = B_{(6\mathrm{d})} \rho^{\widehat{m}} \, . \tag{B.1.14}$$

In 5 dimensions the Symplectic-Majorana condition on spinors is similarly

$$(\rho_{\widehat{m}})^* = B_{(5\mathrm{d})} \rho^{\widehat{m}} \, . \tag{B.1.15}$$

In 4 dimensions the Weyl spinors are irreducible, however 4-dimensional Dirac spinor can obey a Symplectic-Majorana condition identical to (B.1.15).

B.1.3 Spinor Decompositions

<u>6d to 5d :</u>

A Dirac spinor in 6 dimensions decomposes into two 5-dimensional spinors. A 6-dimensional spinor $\underline{\rho} = (\underline{\rho}^{\underline{m}})$ (eight components) of positive chirality reduces to a single 5-dimensional spinor $\rho = (\rho^{m'})$, with the embedding

$$\underline{\rho} = \; \rho \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{B.1.16}$$

For a 6-dimensional spinor of negative chirality, the 5-dimensional spinor is embedded in the complementary four spinor components. The 6-dimensional Symplectic-Majorana condition (B.1.14) on $\underline{\rho}_{\widehat{m}}$ reduces to the 5-dimensional Symplectic-Majorana condition (B.1.15) on $\rho_{\widehat{m}}$ if $\underline{\rho}_{\widehat{m}}$ has positive chirality, or reduces to the opposite reality condition (extra minus sign in the rhs of (B.1.15)) if $\underline{\rho}_{\widehat{m}}$ has negative chirality.

5d to 4d :

A 5-dimensional spinor $\rho = (\rho^{m'})$ decomposes into two 4-dimensional Weyl spinors ψ_+, ψ_- of opposite chiralities, with the embedding

$$\rho = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \psi_+ + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_- \ = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} . \tag{B.1.17}$$

If $\rho^{\widehat{m}}$ obeys the 5-dimensional Symplectic-Majorana condition (B.1.15), the spinors $\psi_+^{\widehat{m}}, \psi_-^{\widehat{m}}$ are not independent. They form four-component spinors which obey a 4-dimensional Symplectic-Majorana condition:

$$\begin{pmatrix} \psi_{-\widehat{m}} \\ \psi_{+\widehat{m}} \end{pmatrix}^* = B_{(4\mathrm{d})} \begin{pmatrix} \psi_-{}^{\widehat{m}} \\ \psi_+{}^{\widehat{m}} \end{pmatrix} . \tag{B.1.18}$$

With these conventions, we obtain for two 5-dimensional spinors $\rho, \tilde{\rho}$ the decomposition of bilinears

$$\begin{aligned} \rho\tilde{\rho} &= \rho_{m'}\tilde{\rho}^{m'} = \psi_{+p}\tilde{\psi}_+^p - \psi_{-\dot{p}}\tilde{\psi}_+^{\dot{p}} = \psi_+\tilde{\psi}_+ - \psi_-\tilde{\psi}_- \,, \\ \rho\gamma^5\tilde{\rho} &= \rho_{m'}(\gamma^5)^{m'}{}_{n'}\tilde{\rho}^{n'} = \psi_{+p}\tilde{\psi}_+^p + \psi_{-\dot{p}}\tilde{\psi}_+^{\dot{p}} = \psi_+\tilde{\psi}_+ + \psi_-\tilde{\psi}_- \\ \rho\gamma^\mu\tilde{\rho} &= \psi_{+p}(\tau^\mu)^p{}_{\dot{p}}\tilde{\psi}_-^{\dot{p}} + \psi_{-\dot{p}}(\bar{\tau}^\mu)^{\dot{p}}{}_p\tilde{\psi}_+^p = \psi_+\tau^\mu\tilde{\psi}_- + \psi_-\bar{\tau}^\mu\tilde{\psi}_+ \,, \end{aligned} \tag{B.1.19}$$

with $(\tau_1, \tau_2, \tau_3, \tau_4) = (-\mathbf{1}_2, \sigma_1, \sigma_2, \sigma_3)$ and $(\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3, \bar{\tau}_4) = (-\mathbf{1}_2, -\sigma_1, -\sigma_2, -\sigma_3)$.

R-symmetry reduction :

In Chapter 6 we considered the reduction of the R-symmetry group

$$\mathfrak{sp}(4)_R \ \to \ \mathfrak{su}(2)_R \oplus \mathfrak{so}(2)_R \,. \tag{B.1.20}$$

The fundamental index $\widehat{m}$ of $^(4)_R$ decomposes into the index $(\widehat{p}, \widehat{x})$ of $\mathfrak{su}(2)_R \oplus \mathfrak{so}(2)_R$. A (collection of) spinors $\rho_{\widehat{m}}$ in any spacetime dimension can be gathered in a column four-vector ρ with each components being a full spinor. The decomposition is then

$$\rho = \ \rho^{(1)} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho^{(2)} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{B.1.21}$$

with $\rho^{(1)} = (\rho^{(1)}{}_{\widehat{p}})$ transforming in the $(\mathbf{2})_{+1}$ of $\mathfrak{su}(2)_R \oplus \mathfrak{so}(2)_R$ and $\rho^{(2)} = (\rho^{(2)}{}_{\widehat{p}})$ transforming in the $(\mathbf{2})_{-1}$. So the four spinors $\rho_{\widehat{m}}$ get replaced by the four spinors $\rho^{(1)}{}_{\widehat{p}}, \rho^{(2)}{}_{\widehat{p}}$. From the $\mathfrak{sp}(4)_R$ invariant tensor $\Omega_{\widehat{m}\widehat{n}}$, with $\Omega = \epsilon \otimes \sigma_1$, and the explicit gamma matrices (B.1.12) we find the bilinear decompositions. For instance

$$\rho^{\widehat{m}}\tilde{\rho}_{\widehat{m}} = \rho^{(1)\widehat{p}}\tilde{\rho}^{(2)}_{\widehat{p}} + \rho^{(2)\widehat{p}}\tilde{\rho}^{(1)}_{\widehat{p}} \,, \tag{B.1.22}$$

$$\rho\Gamma^{\widehat{a}}\tilde{\rho} \equiv \rho^{\widehat{m}}(\Gamma^{\widehat{a}})_{\widehat{m}}{}^{\widehat{n}}\tilde{\rho}_{\widehat{n}} \ = \ \rho^{(2)\widehat{p}}(\sigma^{\widehat{a}})_{\widehat{p}}{}^{\widehat{q}}\tilde{\rho}^{(1)}_{\widehat{q}} - \rho^{(1)\widehat{p}}(\sigma^{\widehat{a}})_{\widehat{p}}{}^{\widehat{q}}\tilde{\rho}^{(2)}_{\widehat{q}} \ \equiv \ \rho^{(2)}\sigma^{\widehat{a}}\tilde{\rho}^{(1)} - \rho^{(1)}\sigma^{\widehat{a}}\tilde{\rho}^{(2)} \,.$$

Furthermore, there is a useful identity

$$(\Gamma^{\widehat{A}})^{\widehat{m}\widehat{n}}(\Gamma_{\widehat{A}})_{\widehat{r}\widehat{s}} = 4\delta^{[\widehat{m}}{}_{\widehat{r}}\delta^{\widehat{n}]}{}_{\widehat{s}} - \Omega^{\widehat{m}\widehat{n}}\Omega_{\widehat{r}\widehat{s}} \ . \tag{B.1.23}$$

B.2 Killing spinors for the S^2 background

In this appendix we determine the solutions to the Killing spinor equations for the S^2 background in section 6.1.2.

B.2.1 $\delta\psi^{\widehat{m}}_{A} = 0$

The supersymmetry transformations are parametrized by two eight-component spinors $\varepsilon^{\widehat{m}}, \eta^{\widehat{m}}$ with an index $\widehat{m}$ transforming in the **4** of $\mathfrak{sp}(4)_R$. The first Killing spinor equation reduces with our ansätze to

$$0 = \delta\psi^{\widehat{m}}_{\underline{A}} = \mathcal{D}_{\underline{A}}\varepsilon^{\widehat{m}} + \frac{1}{24}\,(T^{\widehat{m}\widehat{n}})^{\underline{BCD}}\Gamma_{\underline{BCD}}\Gamma_{\underline{A}}\varepsilon^{\widehat{n}} + \Gamma_{\underline{A}}\eta^{\widehat{m}} \tag{B.2.1}$$

with

$$\begin{aligned} \mathcal{D}_{\underline{\mu}}\varepsilon^{\widehat{m}} &= \partial_{\underline{\mu}}\varepsilon^{\widehat{m}} + \frac{1}{2}b_{\underline{\mu}}\varepsilon^{\widehat{m}} + \frac{1}{4}\tilde{\omega}_{\underline{\mu}}^{\underline{BC}}\Gamma_{\underline{BC}}\varepsilon^{\widehat{m}} - \frac{1}{2}V_{\underline{\mu}}^{\widehat{m}}{}_{\widehat{n}}\varepsilon^{\widehat{n}} \\ \tilde{\omega}_{\underline{\mu}}^{\underline{AB}} &= 2e^{\underline{\nu}[\underline{A}}\partial_{[\underline{\mu}}e_{\underline{\nu}]}{}^{\underline{B}]} - e^{\underline{\rho}[\underline{A}}e^{\underline{B}]\sigma}e_{\underline{\mu}}^{\underline{C}}\partial_{\underline{\rho}}e_{\underline{\sigma C}} + 2e_{\underline{\mu}}^{[\underline{A}}b^{\underline{B}]} \;=\; \omega_{\underline{\mu}}^{\underline{AB}} + 2e_{\underline{\mu}}^{[\underline{A}}b^{\underline{B}]} \ , \end{aligned} \tag{B.2.2}$$

where the background fields have been converted to $\mathfrak{sp}(4)_R$ representations with

$$V_{\underline{A}}^{\widehat{m}}{}_{\widehat{n}} = V_{\underline{A}\widehat{B}\widehat{C}}(\Gamma^{\widehat{B}\widehat{C}})^{\widehat{m}}{}_{\widehat{n}} \quad , \quad T^{\widehat{m}\widehat{n}}_{\underline{BCD}} = T_{\widehat{A}\underline{BCD}}(\Gamma^{\widehat{A}})^{\widehat{m}\widehat{n}} \quad , \quad D^{\widehat{m}\widehat{n}}{}_{\widehat{r}\widehat{s}} = D_{\widehat{A}\widehat{B}}(\Gamma^{\widehat{A}})^{\widehat{m}\widehat{n}}(\Gamma^{\widehat{B}})_{\widehat{r}\widehat{s}} \ . \tag{B.2.3}$$

We choose to set $\eta = 0$. After plugging our ansatz, in particular $T^{\widehat{m}\widehat{n}}_{\underline{BCD}} = b_{\underline{A}} = 0$, we obtain:

$$\begin{aligned} 0 &= \partial_\phi\varepsilon^{\widehat{m}} - \frac{1}{2r}\ell'(\theta)\,\Gamma^{56}\varepsilon^{\widehat{m}} - \frac{1}{2}\,v(\theta)\,(\Gamma^{\widehat{4}\widehat{5}})^{\widehat{m}}{}_{\widehat{n}}\varepsilon^{\widehat{n}} \\ 0 &= \partial_{\mu'}\varepsilon^{\widehat{m}} \quad , \quad \mu' = x^1, x^2, x^3, x^4, \theta \ , \end{aligned} \tag{B.2.4}$$

We find solutions for constant spinors $\varepsilon^{\widehat{m}}$ satisfying:

$$0 \;=\; -\Gamma^{56}\varepsilon^{\widehat{m}} + (\Gamma^{\widehat{4}\widehat{5}})^{\widehat{m}}{}_{\widehat{n}}\varepsilon^{\widehat{n}} \, , \tag{B.2.5}$$

with

$$v(\theta) = -\frac{\ell'(\theta)}{r} \ . \tag{B.2.6}$$

The condition (B.2.5) projects out half of the components of a constant spinor, leaving eight real supercharges in Lorentzian signature, or eight complex supercharges in Euclidean signature.

B.2.2 $\delta\chi_{\widehat{r}}^{\widehat{m}\widehat{n}} = 0$

The second Killing spinor equation is given by

$$\begin{aligned} 0 &= \delta\chi_{\widehat{r}}^{\widehat{m}\widehat{n}} \\ &= \frac{5}{32}\left(\mathcal{D}_{\underline{A}}T^{\widehat{m}\widehat{n}}_{\underline{BCD}}\right)\Gamma^{\underline{BCD}}\Gamma^{\underline{A}}\varepsilon_{\widehat{r}} - \frac{15}{16}\Gamma^{\underline{BC}}R^{[\widehat{m}}_{\underline{BC}\,\widehat{r}}\varepsilon^{\widehat{n}]} - \frac{1}{4}D^{\widehat{m}\widehat{n}}{}_{\widehat{r}\widehat{s}}\varepsilon^{\widehat{s}} + \frac{5}{8}T^{\widehat{m}\widehat{n}}_{\underline{BCD}}\Gamma^{\underline{BCD}}\eta_{\widehat{r}} - \text{traces}\,, \end{aligned} \tag{B.2.7}$$

with

$$\begin{aligned} \mathcal{D}_{\underline{\mu}}T^{\widehat{m}\widehat{n}}_{\underline{BCD}} &= \partial_{\underline{\mu}}T^{\widehat{m}\widehat{n}}_{\underline{BCD}} + 3\tilde{\omega}_{\underline{\mu}[\underline{B}}^{\underline{E}}T^{\widehat{m}\widehat{n}}_{\underline{CD}]\underline{E}} - b_{\underline{\mu}}\,T^{\widehat{m}\widehat{n}}_{\underline{BCD}} + V_{\underline{\mu}\widehat{r}}^{[\widehat{m}}T^{\widehat{n}]\widehat{r}}_{\underline{BCD}} \\ R^{\widehat{m}\widehat{n}}_{\underline{\mu\nu}} &= 2\partial_{[\underline{\mu}}V^{\widehat{m}\widehat{n}}_{\underline{\nu}]} + V_{[\underline{\mu}}^{\widehat{r}(\widehat{m}}V^{\widehat{n})}_{\underline{\mu}]\widehat{r}}\,. \end{aligned} \tag{B.2.8}$$

Here, 'traces' indicates terms proportional to invariant tensors $\Omega^{\widehat{m}\widehat{n}}, \delta^{\widehat{m}}_{\widehat{r}}, \delta^{\widehat{n}}_{\widehat{r}}$. Again the background fields are converted to $\mathfrak{sp}(4)_R$ representations using (B.2.3).

With $T^{\widehat{m}\widehat{n}}_{\underline{BCD}} = 0$, we obtain the simpler conditions

$$0 = -\frac{15}{4}\Gamma^{\underline{BC}}R^{[\widehat{m}}_{\underline{BC}\,\widehat{r}}\varepsilon^{\widehat{n}]} - D^{\widehat{m}\widehat{n}}{}_{\widehat{r}\widehat{s}}\varepsilon^{\widehat{s}} - \text{traces}\,. \tag{B.2.9}$$

The R-symmetry field strength has a single non-vanishing component, corresponding to a flux on S^2:

$$R^{\widehat{m}\widehat{n}}_{\theta\phi} = -R^{\widehat{m}\widehat{n}}_{\phi\theta} = -\frac{\ell''(\theta)}{r}\,(\Gamma^{\widehat{45}})^{\widehat{m}\widehat{n}}\,. \tag{B.2.10}$$

In flat space indices this becomes

$$R^{\widehat{m}\widehat{n}}_{56} = -R^{\widehat{m}\widehat{n}}_{65} = -\frac{\ell''(\theta)}{r^2\ell(\theta)}\,(\Gamma^{\widehat{45}})^{\widehat{m}\widehat{n}}\,. \tag{B.2.11}$$

Moreover our ansatz for $D_{\widehat{A}\widehat{B}}$ (6.1.23) can be re-expressed in $\mathfrak{sp}(4)_R$ indices as:

$$D^{\widehat{m}\widehat{n}}{}_{\widehat{r}\widehat{s}} = d\Big[5(\Gamma^{\widehat{45}})^{[\widehat{m}}{}_{\widehat{r}}(\Gamma^{\widehat{45}})^{\widehat{n}]}{}_{\widehat{s}} - \delta^{[\widehat{m}}{}_{\widehat{r}}\delta^{\widehat{n}]}{}_{\widehat{s}} - \Omega^{\widehat{m}\widehat{n}}\Omega_{\widehat{r}\widehat{s}}\Big]\,, \tag{B.2.12}$$

where the two last terms lead only to "traces" contributions in (B.2.9) and hence drop from the equations. We obtain

$$0 = \frac{15}{2}\frac{\ell''(\theta)}{r^2\ell(\theta)}\,\Gamma^{56}(\Gamma^{\widehat{45}})^{[\widehat{m}}{}_{\widehat{r}}\varepsilon^{\widehat{n}]} - 5d(\Gamma^{\widehat{45}})^{[\widehat{m}}{}_{\widehat{r}}(\Gamma^{\widehat{45}})^{\widehat{n}]}{}_{\widehat{s}}\varepsilon^{\widehat{s}}\,. \tag{B.2.13}$$

Using (B.2.5), we solve the equations without further constraints on $\varepsilon^{\widehat{m}}$ for

$$d = \frac{3}{2}\frac{\ell''(\theta)}{r^2\ell(\theta)}\,. \tag{B.2.14}$$

The background we found corresponds to the twisting $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R \to \mathfrak{u}(1)$ on S^2. It preserves half of the supersymmetry of the flat space theory.

B.3 6d to 5d Reduction for $b_\mu = 0$

In this appendix we detail the reduction of the 6-dimensional equations of motion on an S^1. This is done following ([157, 158]) however we choose to gauge fix $b_{\underline{\mu}} = 0$, which is possible without loss of generality.

We start by decomposing the 6-dimensional frame:

$$e_{\underline{A}}^{\underline{\mu}} = \begin{pmatrix} e_{A'}^{\mu'} & e_{A'}^{\phi} \equiv C_{A'} \\ e_6^{\mu'} \equiv 0 & e_6^{\phi} \equiv \alpha \end{pmatrix}, \tag{B.3.1}$$

where the 5-dimensional indices are primed. We work in the gauge $b_{\underline{\mu}} = 0$, which is achieved by fixing the special conformal generators, $K_{\underline{A}}$. Note that this choice is different from the gauge fixing of $b_{\underline{\mu}}$ in ([157,158]), in particular α is not covariantly constant in this case. Furthermore, we fix the conformal supersymmetry generators to ensure $\psi_5 = 0$, which means that $e_6^{\underline{\mu}} = 0$ is invariant under supersymmetry transformations. For a general background the bosonic supergravity fields descend to 5-dimensional fields as

$$\begin{aligned} D_{\widehat{r}\widehat{s}}^{\widehat{m}\widehat{n}} &\to D_{\widehat{r}\widehat{s}}^{\widehat{m}\widehat{n}} \\ V_{\underline{A}}^{\widehat{m}\widehat{n}} &\to \begin{cases} V_{A'}^{\widehat{m}\widehat{n}} & a \neq 6 \\ S^{\widehat{m}\widehat{n}} & a = 6 \end{cases} \\ T_{\underline{ABC}}^{\widehat{m}\widehat{n}} &\to T_{A'B'6}^{\widehat{m}\widehat{n}} \equiv T_{A'B'}^{\widehat{m}\widehat{n}}\,. \end{aligned} \tag{B.3.2}$$

The components of the spin connection along the ϕ direction are given by

$$w_{\phi}^{A'6} = \frac{1}{\alpha^2} e^{\mu' A'} \partial_{\mu'} \alpha\,, \quad w_{\phi}^{A'b'} = -\frac{1}{\alpha^2} G^{A'B'} \quad w_{\mu'}^{A'6} = \frac{1}{\alpha} e^{\nu' A'} G_{\mu'\nu'}, \tag{B.3.3}$$

where $G = dC$ and can be derived from the 6-dimensional vielbein using

$$\omega_{\underline{\mu}}^{\underline{AB}} = 2e^{\underline{\nu}[\underline{A}} \partial_{[\underline{\mu}} e_{\underline{\nu}]}^{\underline{B}]} - e^{\underline{\rho}[\underline{A}} e^{\underline{B}]\sigma} e_{\underline{\mu}}^{\underline{C}} \partial_{\underline{\rho}} e_{\underline{\sigma C}}. \tag{B.3.4}$$

B.3.1 Equations of Motion for B

In order to reduce the equation of motion for the H field we proceed as in the 6d-4d reduction and decompose the field as:

$$H = \frac{1}{3!} H_{A'B'C'} e^{A'} \wedge e^{B'} \wedge^{C'} + \frac{1}{2} H_{D'E'6} e^{D'} \wedge e^{E'} \wedge e^6. \tag{B.3.5}$$

The 6-dimensional equation of motion reduces to

$$\begin{aligned} dH &= 0 \\ H_{\underline{ABC}}^{-} - \frac{1}{2} \Phi_{\widehat{m}\widehat{n}} T_{\underline{ABC}}^{\widehat{m}\widehat{n}} &= 0\,. \end{aligned} \tag{B.3.6}$$

From this and the second equation in (B.3.6) we can express the two components of H as

$$\begin{aligned} H_{A'B'6} &= \alpha F_{A'B'} \\ H_{A'B'C'} &= \frac{1}{2}\epsilon_{A'B'C'}{}^{D'E'}(\alpha F_{D'E'} - \Phi_{\widehat{m}\widehat{n}} T^{\widehat{m}\widehat{n}}_{D'E'})\,, \end{aligned} \tag{B.3.7}$$

where $F_{\mu'\nu'}$ is a two-form in 5 dimensions. Substituting this into the expansion of H and reducing to 5-dimensional we obtain

$$H = \alpha *_{5d} (F - \frac{1}{\alpha}\Phi_{\widehat{m}\widehat{n}} T^{\widehat{m}\widehat{n}}) + F\wedge C + F\wedge d\varphi\,, \tag{B.3.8}$$

The equation of motion $dH = 0$ implies

$$dF = 0, \quad F\wedge dC + d(\alpha \star_{5d} F - \Phi_{\widehat{m}\widehat{n}} T^{\widehat{m}\widehat{n}}) \tag{B.3.9}$$

which can be integrated to the action in 5d

$$S_F = -\int \mathrm{tr}[\alpha\tilde{F}\wedge *_{5d}\tilde{F} + C\wedge F\wedge F]\,, \tag{B.3.10}$$

where

$$\tilde{F} = F - \frac{1}{\alpha}\Phi_{\widehat{m}\widehat{n}} T^{\widehat{m}\widehat{n}}\,. \tag{B.3.11}$$

Together with the constraint $dF = 0$, which identifies F with the field strength of a 5-dimensional connection A, given by $F_{\mu'\nu'} = \partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'}$.

B.3.2 Equation of Motion for the Scalars

Reducing the equation of motion down to 5-dimensional we find

$$\mathcal{D}^2\Phi^{\widehat{m}\widehat{n}} + 2F_{A'B'}T^{A'B'}_{\widehat{m}\widehat{n}} + (M_\Phi)^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}}\Phi^{\widehat{r}\widehat{s}} = 0\,, \tag{B.3.12}$$

where

$$\begin{aligned} \mathcal{D}_{\mu'}\Phi^{\widehat{m}\widehat{n}} &= \partial_{\mu'} - V^{[\widehat{m}}_{\mu'\widehat{r}}\Phi^{\widehat{n}]\widehat{r}} + \underline{[A_{\mu'}, \Phi^{\widehat{m}\widehat{n}}]} \\ \mathcal{D}^2\Phi^{\widehat{m}\widehat{n}} &= (\partial^{A'} + \omega^{B'A'}_{B'})\mathcal{D}_{A'}\Phi^{\widehat{m}\widehat{n}} - V^{\widehat{[m}}_{\mu'\widehat{r}}\mathcal{D}^{\mu'}\Phi^{\widehat{n}]}_{\widehat{r}} \\ (M_\Phi)^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}} &= -\frac{R_{6d}}{5}\delta^{[\widehat{m}}_{\widehat{r}}\delta^{\widehat{n}]}_{\widehat{s}} + \frac{1}{\alpha}C^{\mu'}\partial_{\mu'}\alpha S^{[\widehat{m}}_{\widehat{r}}\Phi^{\widehat{n}]\widehat{r}} + \frac{1}{2}\alpha^2(S^{[\widehat{m}}_{\widehat{r}}S^{\widehat{n}]}_{\widehat{s}} - S^{\widehat{t}}_{\widehat{s}}S^{[\widehat{m}}_{\widehat{t}}\delta^{\widehat{n}]}_{\widehat{r}}) - \frac{1}{15}D^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}} - T^{A'B'}_{\widehat{r}\widehat{s}}T^{\widehat{m}\widehat{n}}_{A'B'}\,. \end{aligned} \tag{B.3.13}$$

This equation of motion can be integrated to

$$\mathrm{S}_\Phi = -\int d^5x\ \sqrt{|g|}\,\alpha^{-1}\left(\mathcal{D}_{A'}\Phi^{\widehat{m}\widehat{n}}\mathcal{D}^{A'}\Phi_{\widehat{m}\widehat{n}} + 4\Phi^{\widehat{m}\widehat{n}}F_{A'B'}T^{A'B'}_{\widehat{m}\widehat{n}} - \Phi_{\widehat{m}\widehat{n}}(M_\Phi)^{\widehat{m}\widehat{n}}_{\widehat{r}\widehat{s}}\Phi^{\widehat{r}\widehat{s}}\right)\,. \tag{B.3.14}$$

B.3.3 Equation of Motion for the Spinors

We decompose the 6-dimensional field to

$$\rho^{\underline{m}\widehat{m}} \to \begin{pmatrix} 0 \\ i\rho^{m'\widehat{m}} \end{pmatrix} . \tag{B.3.15}$$

Then for a general background the 6-dimensional equation of motion reduces to

$$i\not{D}\rho^{m'\widehat{m}} + (M_\rho)_{n'\widehat{n}}^{m'\widehat{m}}\rho^{n'\widehat{n}} = 0\,, \tag{B.3.16}$$

where

$$\begin{aligned} D_{\mu'}\rho^{m'\widehat{m}} &= \left(\partial_{\mu'} + \frac{1}{4}\omega_{\mu'}^{A'B'}\gamma_{A'B'}\right)\rho^{m'\widehat{m}} - \frac{1}{2}V_{\mu'\widehat{n}}^{\widehat{m}}\rho^{\widehat{n}} \\ (M_\rho)_{n'\widehat{n}}^{m'\widehat{m}} &= \alpha\left(-\frac{1}{2}S_{\widehat{n}}^{\widehat{m}}\delta_{n'}^{m'} + \frac{1}{8\alpha^2}G_{A'B'}(\gamma^{A'B'})_{n'}^{m'}\delta_{\widehat{n}}^{\widehat{m}} - \frac{i}{2\alpha^2}e^{\mu'A'}\partial_{\mu'}\alpha(\gamma_{A'})_{n'}^{m'}\delta_{\widehat{n}}^{\widehat{m}}\right) \\ &\quad + \frac{1}{2\alpha^2}(\gamma^{\mu'}\gamma^{\nu'})_{n'}^{m'}C_{\mu'}\partial_{\nu'}\alpha + \frac{1}{2}T_{A'B'\widehat{n}}^{\widehat{m}}(\gamma^{A'B'})_{n'}^{m'}\,. \end{aligned} \tag{B.3.17}$$

From this we obtain the action

$$S_\rho = -\int d^5x\sqrt{|g|}\,\alpha^{-1}\rho_{m\widehat{m}}\left(i\not{D}_n^m\rho^{n\widehat{m}} + (M_\rho)_{n\widehat{n}}^{m\widehat{m}}\rho^{n\widehat{n}}\right) . \tag{B.3.18}$$

B.4 Supersymmetry Variations of the 5-dimensional Action

The supersymmetry variations (6.2.14), which leave the action S in section 6.2.2 invariant, can be decomposed with respect to the R-symmetry of the 5-dimensional action, following the rules of appendix B.1.3. The scalar and gauge field variations are then

$$\begin{aligned} \delta A_\mu &= -\ell(\theta)\left(\epsilon^{(1)\widehat{p}}\gamma_\mu\rho_{\widehat{p}-}^{(2)} + \epsilon^{(2)\widehat{p}}\gamma_\mu\rho_{\widehat{p}+}^{(1)}\right) \\ \delta A_\theta &= -r\ell(\theta)\left(\epsilon^{(1)\widehat{p}}\rho_{\widehat{p}+}^{(2)} - \epsilon^{(2)\widehat{p}}\rho_{\widehat{p}-}^{(1)}\right) \\ \delta\varphi^{\widehat{a}} &= i\left(\epsilon^{(1)}{}_{\widehat{p}}(\sigma^{\widehat{a}})^{\widehat{p}\widehat{q}}\rho_{\widehat{q}+}^{(2)} - \epsilon^{(2)}{}_{\widehat{p}}(\sigma^{\widehat{a}})^{\widehat{p}\widehat{q}}\rho_{\widehat{q}-}^{(1)}\right) \\ \delta\varphi &= -2\epsilon^{(1)\widehat{p}}\rho_{\widehat{p}+}^{(1)}\,, \quad \delta\bar{\varphi} \ = \ 2\epsilon^{(2)\widehat{p}}\rho_{\widehat{p}-}^{(2)} \end{aligned} \tag{B.4.1}$$

and for the fermions

$$\begin{aligned}
\delta\rho^{(1)}_{\widehat{p}+} &= \frac{i}{8\ell(\theta)} F_{\mu\nu}\gamma^{\mu\nu}\epsilon^{(1)}_{\widehat{p}} - \frac{i}{4}\mathcal{D}_\mu\varphi\gamma^\mu\epsilon^{(2)}_{\widehat{p}} + \frac{1}{4r}\mathcal{D}_\theta\varphi_{\widehat{p}}^{\widehat{q}}\epsilon^{(1)}_{\widehat{q}} - \frac{\ell(\theta)}{8}\left(\epsilon^{\widehat{a}\widehat{b}\widehat{c}}[\varphi_{\widehat{a}},\varphi_{\widehat{b}}](\sigma_{\widehat{c}})_{\widehat{p}}^{\widehat{q}}\epsilon^{(1)}_{\widehat{q}} - i[\varphi,\bar{\varphi}]\epsilon^{(1)}_{\widehat{p}}\right)\\
\delta\rho^{(1)}_{\widehat{p}-} &= \frac{i}{4r\ell(\theta)} F_{\mu\theta}\gamma^{\mu}\epsilon^{(1)}_{\widehat{p}} + \frac{1}{4}\mathcal{D}_\mu\varphi_{\widehat{p}}^{\widehat{q}}\gamma^\mu\epsilon^{(1)}_{\widehat{q}} + \frac{i}{4r}\left(\mathcal{D}_\theta\varphi - \frac{\ell'(\theta)}{r\ell(\theta)}\varphi\right)\epsilon^{(2)}_{\widehat{p}} - \frac{\ell(\theta)}{4}[\varphi,\varphi_{\widehat{p}}^{\widehat{q}}]\epsilon^{(2)}_{\widehat{q}}\\
\delta\rho^{(2)}_{\widehat{p}+} &= -\frac{i}{4r\ell(\theta)} F_{\mu\theta}\gamma^{\mu}\epsilon^{(2)}_{\widehat{p}} - \frac{1}{4}\mathcal{D}_\mu\varphi_{\widehat{p}}^{\widehat{q}}\gamma^\mu\epsilon^{(2)}_{\widehat{q}} - \frac{i}{4r}\left(\mathcal{D}_\theta\bar{\varphi} - \frac{\ell'(\theta)}{r\ell(\theta)}\bar{\varphi}\right)\epsilon^{(1)}_{\widehat{p}} - \frac{\ell(\theta)}{4}[\bar{\varphi},\varphi_{\widehat{p}}^{\widehat{q}}]\epsilon^{(1)}_{\widehat{q}}\\
\delta\rho^{(2)}_{\widehat{p}-} &= \frac{i}{8\ell(\theta)} F_{\mu\nu}\gamma^{\mu\nu}\epsilon^{(2)}_{\widehat{p}} + \frac{i}{4}\mathcal{D}_\mu\bar{\varphi}\gamma^\mu\epsilon^{(1)}_{\widehat{p}} + \frac{1}{4r}\mathcal{D}_\theta\varphi_{\widehat{p}}^{\widehat{q}}\epsilon^{(2)}_{\widehat{q}} - \frac{\ell(\theta)}{8}\left(\epsilon^{\widehat{a}\widehat{b}\widehat{c}}[\varphi_{\widehat{a}},\varphi_{\widehat{b}}](\sigma_{\widehat{c}})_{\widehat{p}}^{\widehat{q}}\epsilon^{(2)}_{\widehat{q}} + i[\varphi,\bar{\varphi}]\epsilon^{(2)}_{\widehat{p}}\right),
\end{aligned} \tag{B.4.2}$$

where $\varphi_{\widehat{p}}^{\widehat{q}} = \sum_{\widehat{a}}\varphi^{\widehat{a}}(\sigma^{\widehat{a}})_{\widehat{p}}^{\widehat{q}}$.

B.5 Aspects of the 4-dimensional Sigma-Model

In this appendix we summarize properties of the sigma-model defined in section 6.3 and provide details on integrating out the gauge field and the scalars φ and $\bar{\varphi}$.

B.5.1 Useful Relations

The three symplectic structures of the hyper-Kähler target can be used to define the three complex structures $\omega^{\widehat{a}}_K{}^I = \omega^{\widehat{a}}_{KJ}G^{JI}$, which satisfy

$$\omega_{\widehat{a}I}{}^J\omega_{\widehat{b}J}{}^K = -\delta_{\widehat{a}\widehat{b}}\delta_I^K + \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\omega_I^{\widehat{c}K}\,. \tag{B.5.1}$$

The complex structures exchange $\Upsilon_I^{\widehat{a}}$ and $\Upsilon_I^{(\theta)}$ in the following fashion

$$\begin{aligned}
\omega_I^{\widehat{a}J}\Upsilon_J^{(\theta)} &= -\Upsilon_I^{\widehat{a}}\\
\omega_I^{\widehat{a}J}\Upsilon_J^{\widehat{b}} &= \delta^{\widehat{a}\widehat{b}}\Upsilon_I^{(\theta)} + \epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Upsilon_{I\widehat{c}}\,.
\end{aligned} \tag{B.5.2}$$

Here, we made use of the completeness relations ([173])

$$\begin{aligned}
G^{IJ}\Upsilon_I^{\widehat{a}\alpha}(\theta)\Upsilon_J^{\widehat{b}\beta}(\tau) + \sum_i \Psi_i^{\widehat{a}a\alpha}(\theta)\Psi_i^{\widehat{b}\beta}(\tau) &= \delta^{\widehat{a}\widehat{b}}\delta^{\alpha\beta}\,\delta(\theta-\tau)\\
G^{IJ}\Upsilon_I^{(\theta)a\alpha}(\theta)\Upsilon_J^{(\theta)\beta}(\tau) + \sum_i \Psi_i^{(\theta)a\alpha}(\theta)\Psi_i^{(\theta)\beta}(\tau) &= \delta^{\alpha\beta}\delta(\theta-\tau)\\
G^{IJ}\Upsilon_I^{\widehat{a}\alpha}(\theta)\Upsilon_J^{(\theta)\beta}(\tau) + \sum_i \Psi_i^{\widehat{\alpha}}(\theta)\Psi_i^{(\theta)\beta}(\tau) &= 0,
\end{aligned} \tag{B.5.3}$$

Here α,β are indices labelling generators of the gauge algebra. These functions satisfy the orthogonality conditions

$$\int d\theta\Upsilon_I^{\widehat{a}\alpha}(\theta)\Psi_i^{\widehat{b}\beta}(\theta) = 0\,, \quad \int d\theta\Upsilon_I^{(\theta)\alpha}(\theta)\Psi_i^{(\theta)\beta}(\theta) = 0. \tag{B.5.4}$$

B.5.2 Integrating out Fields

In this appendix we discuss how the scalars $\varphi, \bar{\varphi}$ and the 4-dimensional gauge field A_μ are integrated out in the sigma-model reduction. The equations of motion for $\varphi, \bar{\varphi}$ and A_μ are derived from the action (6.2.36)

$$\begin{aligned}
\mathcal{D}^2\varphi + [\varphi_{\widehat{a}}, [\varphi^{\widehat{a}}, \varphi]] &= -4ir[\rho^{(1)}_{-\widehat{p}}, \rho^{(1)\widehat{p}}_{+}] \\
\mathcal{D}^2\bar{\varphi} + [\varphi_{\widehat{a}}, [\varphi^{\widehat{a}}, \bar{\varphi}]] &= 4ir[\rho^{(2)}_{+\widehat{p}}, \rho^{(2)\widehat{p}}_{-}] \\
\mathcal{D}^2_\theta A_\mu + \left[\varphi_{\widehat{a}}, \left[\varphi^{\widehat{a}}, A_\mu\right]\right] &= [A_\theta, \partial_I A_\theta]\, \partial_\mu X^I + \left[\varphi_{\widehat{a}}, \partial_I \varphi^{\widehat{a}}\right] \partial_\mu X^I - 4i[\rho^{(1)}_{-\widehat{p}}, \gamma_\mu \rho^{(2)\widehat{p}}_{+}]\,,
\end{aligned} \tag{B.5.5}$$

where in the equation of motion for A_μ we made use of the expansion (6.3.19) and (6.3.20). We adopt a convenient gauge for the connection E_I

$$\mathcal{D}_\theta \Upsilon^\theta_I + [\varphi_{\widehat{a}}, \Upsilon^{\widehat{a}}_I] = 0, \tag{B.5.6}$$

which can be re-expressed as

$$\mathcal{D}^2_\theta E_I + [\varphi_{\widehat{a}}, [\varphi^{\widehat{a}}, E_I]] = [A_\theta, \partial_I A_\theta] + [\varphi_{\widehat{a}}, \partial_I \varphi^{\widehat{a}}]\,, \tag{B.5.7}$$

where we have used the gauge fixing condition $\partial_\theta A_\theta = 0$. We evaluate the spinors bilinear in (B.5.5) to give

$$\begin{aligned}
\left[\rho^{(1)}_{-\widehat{p}}, \rho^{(1)\widehat{p}}_{-}\right] &= -4\left(\left[\Upsilon^{\widehat{a}}_I, \Upsilon_{J\widehat{a}}\right] + \left[\Upsilon^{(\theta)}_I, \Upsilon^{(\theta)}_J\right]\right) \lambda^{(1)I}_{\widehat{p}} \lambda^{(1)J\widehat{p}} \\
\left[\rho^{(2)}_{+\widehat{p}}, \rho^{(2)\widehat{p}}_{+}\right] &= -4\left(\left[\Upsilon^{\widehat{a}}_I, \Upsilon_{J\widehat{a}}\right] + \left[\Upsilon^{(\theta)}_I, \Upsilon^{(\theta)}_J\right]\right) \lambda^{(2)I}_{\widehat{p}} \lambda^{(2)J\widehat{p}} \\
\left[\rho^{(1)}_{-\widehat{p}}, \rho^{(2)\widehat{p}}_{+}\right] &= -4\left(\left[\Upsilon^{\widehat{a}}_I, \Upsilon_{J\widehat{a}}\right] + \left[\Upsilon^{(\theta)}_I, \Upsilon^{(\theta)}_J\right]\right) \lambda^{(1)I}_{\widehat{p}} \lambda^{(2)J\widehat{p}}
\end{aligned} \tag{B.5.8}$$

We note that the curvature

$$\Phi_{IJ} = [\nabla_I, \nabla_J]\,, \tag{B.5.9}$$

where $\nabla_I = \partial_I + [E_I, \cdot\,]$, satisfies the equation

$$\mathcal{D}^2_\theta \Phi_{IJ} + [\varphi_{\widehat{a}}, [\varphi^{\widehat{a}}, \Phi_{IJ}]] = 2\left([\Upsilon_{I\widehat{a}}, \Upsilon^{\widehat{a}}_J] + [\Upsilon^{(\theta)}_I, \Upsilon^{(\theta)}_J]\right)\,. \tag{B.5.10}$$

It can be used to solve the equations of motion by

$$\begin{aligned}
\varphi &= 8ir\Phi_{IJ}\lambda^{(1)I}_{\widehat{p}}\lambda^{(1)J\widehat{p}} \\
\bar{\varphi} &= -8ir\Phi_{IJ}\lambda^{(2)I}_{\widehat{p}}\lambda^{(2)J\widehat{p}} \\
A_\mu &= E_I \partial_\mu X^I + 8i\Phi_{IJ}\lambda^{(1)I}_{\widehat{p}}\gamma_\mu \lambda^{(2)J\widehat{p}}.
\end{aligned} \tag{B.5.11}$$

Inserting this back in the action the terms with $\varphi, \bar{\varphi}$ result in

$$S_{\varphi,\bar{\varphi}} = \frac{16}{r\ell}\int d\theta d^4x\sqrt{|g_4|}\mathrm{Tr}(\mathcal{D}_\theta\Phi_{IJ}\mathcal{D}_\theta\Phi_{KL} + [\Phi_{IJ},\varphi^{\widehat{a}}][\Phi_{KL},\varphi_{\widehat{a}}])\lambda^{(1)I\widehat{p}}\lambda^{(1)J}_{\widehat{p}}\lambda^{(2)K\widehat{q}}\lambda^{(2)L}_{\widehat{q}} \tag{B.5.12}$$

Integrating out the gauge filed we obtain three types of terms. The first type are terms such that X^I appear quadratically

$$\begin{aligned} S_{A_\mu,type1} =& -\frac{1}{4r\ell}\int d^4xd\theta\mathrm{Tr}\Big(\mathcal{D}_\theta E_I\mathcal{D}_\theta E_J - 2\partial_I A_\theta\mathcal{D}_\theta E_J + 2\partial_I\varphi^{\widehat{a}}[E_J,\varphi_{\widehat{a}}] \\ &+[E_I,\varphi^{\widehat{a}}][E_J,\varphi_{\widehat{a}}]\Big)\,\partial_\mu X^I\partial^\mu X^J\,. \end{aligned} \tag{B.5.13}$$

These terms combine with terms in the scalar action (6.3.25) to give the sigma-model kinetic term

$$S_{\mathrm{scalars}} + S_{A_\mu,type1} = \frac{1}{4r\ell}\int d^4x\sqrt{|g_4|}G_{IJ}\partial_\mu X^I\partial^\mu X^J\,, \tag{B.5.14}$$

Terms of the second type are linear in X^I and covariantise the kinetic terms of the spinor

$$S_{A\mu,type2} = -\frac{4i}{r\ell}\int d^4xd\theta\sqrt{|g_4|}\mathrm{Tr}(2\Upsilon^{\widehat{a}}_I[E_J,\Upsilon_{K\widehat{a}}] + 2\Upsilon^{(\theta)}_I[E_J,\Upsilon^{(\theta)}_K])\lambda^{(1)I\widehat{p}}\gamma^\mu\lambda^{(2)K}_{\widehat{p}}\partial_\mu X^J. \tag{B.5.15}$$

The terms involving the connection E_I are promoted to covariant derivatives ∇_I when combined with the terms in the spinor action (6.3.28). Using the identities

$$\begin{aligned} \nabla_I\Upsilon^{\widehat{a}}_J &= \Gamma^K_{IJ}\Upsilon^{\widehat{a}}_K + \frac{1}{2}[\Phi_{IJ},\varphi^{\widehat{a}}] \\ \nabla_I\Upsilon^{(\theta)}_J &= \Gamma^K_{IJ}\Upsilon^{(\theta)}_K - \frac{1}{2}\mathcal{D}_\theta\Phi_{IJ}\,, \end{aligned} \tag{B.5.16}$$

where

$$\Gamma_{IJ,K} = -\int d\theta\,\mathrm{Tr}\left(\Upsilon^{\widehat{a}}_K\nabla_{(I}\Upsilon_{J)\widehat{a}} + \Upsilon^{(\theta)}_K\nabla_{(I}\Upsilon^{(\theta)}_{J)}\right)\,, \tag{B.5.17}$$

the kinetic term in the spinor action is covariantised. Lastly, the terms of type 3 give rise to the quartic fermion interaction. Using (B.5.10) these terms simplify to

$$\begin{aligned} S_{A_\mu,type3} =& -\frac{16}{r\ell}\int d^4xd\theta\sqrt{|g|}\mathrm{Tr}(\mathcal{D}_\theta\Phi_{IJ}\mathcal{D}_\theta\Phi_{KL} + [\Phi_{IJ},\varphi^{\widehat{a}}][\Phi_{KL},\varphi_{\widehat{a}}]) \\ &\times(\lambda^{(1)I\widehat{p}}\gamma^\mu\lambda^{(2)J}_{\widehat{p}})(\lambda^{(1)K\widehat{q}}\gamma_\mu\lambda^{(2)L}_{\widehat{q}}). \end{aligned} \tag{B.5.18}$$

Using various identities, including Fierz-type identities,

$$\begin{aligned} (\lambda^{(1)\widehat{p}[I}\lambda^{(1)J]}_{\widehat{p}})(\lambda^{(2)\widehat{q}[K}\lambda^{(2)L]}_{\widehat{q}}) &= 2(\lambda^{(1)\widehat{p}[I}\lambda^{(1)J]\widehat{q}})(\lambda^{(2)[K}_{\widehat{p}}\lambda^{(2)L]}_{\widehat{q}}) \\ \omega^{\widehat{a}}{}_I{}^K\nabla_{[K}\Upsilon^\theta_{J]} &= \nabla_{[I}\Upsilon^{\widehat{a}}_{J]} \\ \nabla_{[I}\Upsilon^{\widehat{a}}_{J]}\lambda^{(i)J}_{\widehat{p}} &= i\nabla_{[I}\Upsilon^\theta_{J]}(\sigma^{\widehat{a}})^{\widehat{q}}_{\widehat{p}}\lambda^{(i)J}_{\widehat{q}} \\ \nabla_{[I}\Upsilon^{\widehat{a}}_{J]}\nabla_{[K}\Upsilon_{L]\widehat{a}}\lambda^{(i)J}_{\widehat{p}}\lambda^{(i)L}_{\widehat{q}} &= 3\nabla_{[I}\Upsilon^\theta_{J]}\nabla_{[K}\Upsilon^\theta_{L]}\lambda^{(i)[J}_{\widehat{p}}\lambda^{(i)L]}_{\widehat{q}} \end{aligned} \tag{B.5.19}$$

it can be shown that this quartic fermion interaction combines with the term (B.5.12) to make the Riemann tensor of the target space appear

$$S_{A_\mu,\text{type3}} + S_{\varphi,\bar{\varphi}} = -\frac{32}{rl}\int d^4x\sqrt{|g_4|}R_{IJKL}(\lambda^{(1)I\widehat{p}}\lambda^{(1)J}_{\widehat{p}})(\lambda^{(2)K\widehat{q}}\lambda^{(2)L}_{\widehat{q}}), \tag{B.5.20}$$

where the Riemann tensor is given by

$$\begin{aligned} R_{IJKL} &= -\int d\theta\,\mathrm{Tr}(2\nabla_{[I}\Upsilon^{\widehat{a}}_{J]}\nabla_{[K}\Upsilon_{L]\widehat{a}} + \nabla_{[I}\Upsilon^{\widehat{a}}_{K]}\nabla_{[J}\Upsilon_{L]\widehat{a}} - \nabla_{[I}\Upsilon^{\widehat{a}}_{L]}\nabla_{[J}\Upsilon_{K]\widehat{a}}) \\ &\quad + 2\nabla_{[I}\Upsilon^{(\theta)}_{J]}\nabla_{[K}\Upsilon^{(\theta)}_{L]} + \nabla_{[I}\Upsilon^{(\theta)}_{K]}\nabla_{[J}\Upsilon^{(\theta)}_{L]} - \nabla_{[I}\Upsilon^{(\theta)}_{L]}\nabla_{[J}\Upsilon^{(\theta)}_{K]}) \\ &= -\frac{1}{4}\int d\theta\,\mathrm{Tr}(2\mathcal{D}_\theta\Phi_{IJ}\mathcal{D}_\theta\Phi_{KL} + 2[\Phi_{IJ},\varphi^{\widehat{a}}][\Phi_{KL},\varphi_{\widehat{a}}] + \mathcal{D}_\theta\Phi_{IK}\mathcal{D}_\theta\Phi_{JL} \\ &\quad + [\Phi_{IK},\varphi^{\widehat{a}}][\Phi_{JL},\varphi_{\widehat{a}}] - \mathcal{D}_\theta\Phi_{IL}\mathcal{D}_\theta\Phi_{JK} - [\Phi_{IL},\varphi^{\widehat{a}}][\Phi_{JK},\varphi_{\widehat{a}}])\,, \end{aligned} \tag{B.5.21}$$

Combining all the terms we obtain the final sigma model (6.3.29).

Bibliography

[1] J. Borchmann, C. Mayrhofer, E. Palti, and T. Weigand, *SU(5) Tops with Multiple U(1)s in F-theory*, *Nucl.Phys.* **B882** (2014) 1–69, [1307.2902]. (document), 2.9, 3, 3.1, 3.1, 3.2.1, 3.2.3, 3.4, 3.4.1, 3.4.1, 3.6, 3.4.2, 3.7, 3.4.2, A.4

[2] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B443** (1995) 85–126, [hep-th/9503124]. 1, 4, 4.1

[3] J. Bagger and N. Lambert, *Modeling Multiple M2's*, *Phys. Rev.* **D75** (2007) 045020, [hep-th/0611108]. 1, 4, 4.4, 4.4, 4.4

[4] A. Gustavsson, *Algebraic structures on parallel M2-branes*, *Nucl. Phys.* **B811** (2009) 66–76, [0709.1260]. 1, 4, 5, 5.3

[5] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **0810** (2008) 091, [0806.1218]. 1, 4, 5

[6] N. Lambert and C. Papageorgakis, *Nonabelian (2,0) Tensor Multiplets and 3-algebras*, *JHEP* **08** (2010) 083, [1007.2982]. 1, 4.5, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 5, 5.1, 5.1.1, 5.1.2, 5.3, 7

[7] C. Vafa, *Evidence for F-Theory*, *Nucl. Phys.* **B469** (1996) 403–418, [hep-th/9602022]. 1, 2.2

[8] D. R. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi–Yau Threefolds – I*, *Nucl. Phys.* **B473** (1996) 74–92, [hep-th/9602114]. 1, 2.2, 3

[9] D. R. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi–Yau Threefolds – II*, *Nucl. Phys.* **B476** (1996) 437–469, [hep-th/9603161]. 1, 2.2, 3

[10] C. Lawrie and D. Sacco, *Tate's algorithm for F-theory GUTs with two U(1)s*, *JHEP* **03** (2015) 055, [1412.4125]. 1

[11] N. Lambert and D. Sacco, *M2-branes and the (2, 0) superalgebra*, *JHEP* **09** (2016) 107, [1608.04748]. 1

[12] B. Assel, S. Schafer-Nameki, and J.-M. Wong, *M5-branes on S2 x M4: Nahm's Equations and 4d Topological Sigma-models*, 1604.03606. 1, 6

[13] J. Polchinski, *String Theory: Volume 2, Superstring Theory and Beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998. 2.1

[14] T. Weigand, *Lectures on F-theory compactifications and model building*, *Class.Quant.Grav.* **27** (2010) 214004, [1009.3497]. 2.1, 2.2, 2.8, 1

[15] J. H. Silverman, *The arithmetic of elliptic curves*, vol. 106 of *Graduate Texts in Mathematics.* Springer, Dordrecht, second ed., 2009. 2.3, 2.5, A.4

[16] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley Classics Library. Wiley, 2011. 2.6

[17] K. Kodaira, *On compact analytic surfaces*, *Annals of Math.* **77** (1963). 2.7

[18] M. Esole and S.-T. Yau, *Small resolutions of SU(5)-models in F-theory*, *Adv.Theor.Math.Phys.* **17** (2013) 1195–1253, [1107.0733]. 2.7, 3.2.3

[19] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, in *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pp. 33–52. Lecture Notes in Math., Vol. 476. Springer, Berlin, 1975. 2.7, 3

[20] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, *et. al.*, *Geometric singularities and enhanced gauge symmetries*, *Nucl.Phys.* **B481** (1996) 215–252, [hep-th/9605200]. 2.7, 3, 3.3

[21] S. Katz, D. R. Morrison, S. Schafer-Nameki, and J. Sully, *Tate's algorithm and F-theory*, *JHEP* **1108** (2011) 094, [1106.3854]. 2.7, 3, 3.1, 3.2.3, 3.3, A.1

[22] M. Kuntzler and S. Schafer-Nameki, *Tate Trees for Elliptic Fibrations with Rank one Mordell-Weil group*, 1406.5174. 2.7, 2.9, 3, 3.2.2, A.1

[23] D. R. Morrison and D. S. Park, *F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds*, *JHEP* **1210** (2012) 128, [1208.2695]. 2.9, 2.9, 3, 3.2.1, 3.2.3

[24] J. Borchmann, C. Mayrhofer, E. Palti, and T. Weigand, *Elliptic fibrations for $SU(5) \times U(1) \times U(1)$ F-theory vacua*, *Phys.Rev.* **D88** (2013), no. 4 046005, [1303.5054]. 2.9, 3, 3.1, 3.1, 3.2.1, 3.2.3, 3.4.1, A.4

[25] M. Cvetic, D. Klevers, and H. Piragua, *F-Theory Compactifications with Multiple U(1)-Factors: Constructing Elliptic Fibrations with Rational Sections*, *JHEP* **1306** (2013) 067, [1303.6970]. 2.9, 3, 3.1, 3.2.1, 3.2.3, A.4

[26] M. Cvetic, D. Klevers, and H. Piragua, *F-Theory Compactifications with Multiple U(1)-Factors: Addendum*, *JHEP* **1312** (2013) 056, [1307.6425]. 2.9, 3, 3.1, 3.2.1, 3.2.3, A.4

[27] R. Donagi and M. Wijnholt, *Model Building with F-Theory*, *Adv.Theor.Math.Phys.* **15** (2011) 1237–1318, [0802.2969]. 3

[28] C. Beasley, J. J. Heckman, and C. Vafa, *GUTs and Exceptional Branes in F-theory - I*, *JHEP* **01** (2009) 058, [0802.3391]. 3

[29] C. Beasley, J. J. Heckman, and C. Vafa, *GUTs and Exceptional Branes in F-theory - II: Experimental Predictions*, *JHEP* **01** (2009) 059, [0806.0102]. 3

[30] J. J. Heckman, *Particle Physics Implications of F-theory*, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 237–265, [1001.0577]. 1

[31] A. Maharana and E. Palti, *Models of Particle Physics from Type IIB String Theory and F-theory: A Review*, *Int. J. Mod. Phys.* **A28** (2013) 1330005, [1212.0555]. 1

[32] M. Berasaluce-Gonzalez, L. E. Ibanez, P. Soler, and A. M. Uranga, *Discrete gauge symmetries in D-brane models*, *JHEP* **12** (2011) 113, [1106.4169]. 2

[33] M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano, and A. M. Uranga, *Zp charged branes in flux compactifications*, *JHEP* **04** (2013) 138, [1211.5317]. 2

[34] C. Mayrhofer, E. Palti, O. Till, and T. Weigand, *Discrete Gauge Symmetries by Higgsing in four-dimensional F-Theory Compactifications*, *JHEP* **12** (2014) 068, [1408.6831]. 2

[35] I. García-Etxebarria, T. W. Grimm, and J. Keitel, *Yukawas and discrete symmetries in F-theory compactifications without section*, *JHEP* **11** (2014) 125, [1408.6448]. 2

[36] C. Mayrhofer, E. Palti, O. Till, and T. Weigand, *On Discrete Symmetries and Torsion Homology in F-Theory*, *JHEP* **06** (2015) 029, [1410.7814]. 2

[37] H. Hayashi, R. Tatar, Y. Toda, T. Watari, and M. Yamazaki, *New Aspects of Heterotic–F Theory Duality*, *Nucl. Phys.* **B806** (2009) 224–299, [0805.1057]. 3

[38] R. Donagi and M. Wijnholt, *Higgs Bundles and UV Completion in F-Theory*, *Commun.Math.Phys.* **326** (2014) 287–327, [0904.1218]. 3

[39] J. Marsano, N. Saulina, and S. Schafer-Nameki, *Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs*, *JHEP* **08** (2009) 046, [0906.4672]. 3

[40] J. Marsano, N. Saulina, and S. Schafer-Nameki, *Compact F-theory GUTs with $U(1)_{PQ}$*, *JHEP* **04** (2010) 095, [0912.0272]. 3

[41] E. Dudas and E. Palti, *Froggatt-Nielsen models from E(8) in F-theory GUTs*, *JHEP* **1001** (2010) 127, [0912.0853]. 3

[42] H. Hayashi, T. Kawano, Y. Tsuchiya, and T. Watari, *More on Dimension-4 Proton Decay Problem in F-theory – Spectral Surface, Discriminant Locus and Monodromy*, *Nucl.Phys.* **B840** (2010) 304–348, [1004.3870]. 3

[43] E. Dudas and E. Palti, *On hypercharge flux and exotics in F-theory GUTs*, *JHEP* **09** (2010) 013, [1007.1297]. 3

[44] M. J. Dolan, J. Marsano, N. Saulina, and S. Schafer-Nameki, *F-theory GUTs with U(1) Symmetries: Generalities and Survey*, *Phys.Rev.* **D84** (2011) 066008, [1102.0290]. 3

[45] T. W. Grimm and T. Weigand, *On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs*, *Phys.Rev.* **D82** (2010) 086009, [1006.0226]. 3

[46] T. W. Grimm, M. Kerstan, E. Palti, and T. Weigand, *Massive Abelian Gauge Symmetries and Fluxes in F-theory*, *JHEP* **12** (2011) 004, [1107.3842]. 3

[47] A. P. Braun, A. Collinucci, and R. Valandro, *G-flux in F-theory and algebraic cycles*, *Nucl.Phys.* **B856** (2012) 129–179, [1107.5337]. 55 pages, 1 figure/ added refs, corrected typos. 3

[48] T. W. Grimm and H. Hayashi, *F-theory fluxes, Chirality and Chern-Simons theories*, *JHEP* **1203** (2012) 027, [1111.1232]. 53 pages, 5 figures/ v2: typos corrected, minor improvements. 3, 3.2.3

[49] S. Krause, C. Mayrhofer, and T. Weigand, *Gauge Fluxes in F-theory and Type IIB Orientifolds*, *JHEP* **1208** (2012) 119, [1202.3138]. 3

[50] M. Cvetic, T. W. Grimm, and D. Klevers, *Anomaly Cancellation And Abelian Gauge Symmetries In F-theory*, *JHEP* **1302** (2013) 101, [1210.6034]. 3

[51] C. Mayrhofer, E. Palti, and T. Weigand, *U(1) symmetries in F-theory GUTs with multiple sections*, *JHEP* **1303** (2013) 098, [1211.6742]. 3, 3.1, 3.4.1, 3.4.2

[52] V. Braun, T. W. Grimm, and J. Keitel, *New Global F-theory GUTs with U(1) symmetries*, *JHEP* **1309** (2013) 154, [1302.1854]. 3

[53] S. Krippendorf, D. K. Mayorga Pena, P.-K. Oehlmann, and F. Ruehle, *Rational F-Theory GUTs without exotics*, *JHEP* **1407** (2014) 013, [1401.5084]. 3, 3.4

[54] V. Braun, T. W. Grimm, and J. Keitel, *Geometric Engineering in Toric F-Theory and GUTs with U(1) Gauge Factors*, *JHEP* **1312** (2013) 069, [1306.0577]. 3, 3.1, 3.4.1

[55] M. Cvetic, A. Grassi, D. Klevers, and H. Piragua, *Chiral Four-Dimensional F-Theory Compactifications With SU(5) and Multiple U(1)-Factors*, *JHEP* **1404** (2014) 010, [1306.3987]. 3

[56] M. Cvetic, D. Klevers, H. Piragua, and P. Song, *Elliptic fibrations with rank three Mordell-Weil group: F-theory with U(1) x U(1) x U(1) gauge symmetry*, *JHEP* **1403** (2014) 021, [1310.0463]. 3

[57] D. Klevers, D. K. Mayorga Pena, P.-K. Oehlmann, H. Piragua, and J. Reuter, *F-Theory on all Toric Hypersurface Fibrations and its Higgs Branches*, 1408.4808. 3

[58] V. Braun, T. W. Grimm, and J. Keitel, *Complete Intersection Fibers in F-Theory*, 1411.2615. 3

[59] P. Candelas and A. Font, *Duality between the webs of heterotic and type II vacua*, *Nucl. Phys.* **B511** (1998) 295–325, [hep-th/9603170]. 3, 3.1, 3.4.1

[60] K. Kodaira, *On compact complex analytic surfaces. I*, *Ann. of Math. (2)* **71** (1960) 111–152. 3

[61] K. Kodaira, *On compact analytic surfaces. II, III*, *Ann. of Math. (2) 77 (1963), 563–626; ibid.* **78** (1963) 1–40. 3

[62] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, *Inst. Hautes Études Sci. Publ.Math. No.* **21** (1964) 128. 3

[63] L. Lin and T. Weigand, *Towards the Standard Model in F-theory*, 1406.6071. 3

[64] A. Grassi, J. Halverson, J. Shaneson, and W. Taylor, *Non-Higgsable QCD and the Standard Model Spectrum in F-theory*, *JHEP* **01** (2015) 086, [1409.8295]. 3

[65] H. Hayashi, C. Lawrie, and S. Schafer-Nameki, *Phases, Flops and F-theory: SU(5) Gauge Theories*, *JHEP* **1310** (2013) 046, [1304.1678]. 3, 3.2.3

[66] H. Hayashi, C. Lawrie, D. R. Morrison, and S. Schafer-Nameki, *Box Graphs and Singular Fibers*, *JHEP* **1405** (2014) 048, [1402.2653]. 3

[67] M. Esole, S.-H. Shao, and S.-T. Yau, *Singularities and Gauge Theory Phases*, 1402.6331. 3

[68] A. P. Braun and S. Schafer-Nameki, *Box Graphs and Resolutions I*, 1407.3520. 3, 3.2.3

[69] M. Esole, S.-H. Shao, and S.-T. Yau, *Singularities and Gauge Theory Phases II*, 1407.1867. 3, 3.2.3

[70] D. R. Morrison and W. Taylor, *Sections, multisections, and U(1) fields in F-theory*, 1404.1527. 3

[71] V. Braun and D. R. Morrison, *F-theory on Genus-One Fibrations*, *JHEP* **08** (2014) 132, [1401.7844]. 3

[72] L. B. Anderson, I. García-Etxebarria, T. W. Grimm, and J. Keitel, *Physics of F-theory compactifications without section*, *JHEP* **12** (2014) 156, [1406.5180]. 3

[73] M. Auslander and D. A. Buchsbaum, *Unique factorization in regular local rings*, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959) 733–734. 3.1, A.1

[74] C. Lawrie and S. Schäfer-Nameki, *The Tate Form on Steroids: Resolution and Higher Codimension Fibers*, *JHEP* **1304** (2013) 061, [1212.2949]. 3, 3.2.3, 3.2.3

[75] T. W. Grimm, S. Krause, and T. Weigand, *F-Theory GUT Vacua on Compact Calabi-Yau Fourfolds*, *JHEP* **1007** (2010) 037, [0912.3524]. 3.2.3

[76] S. Krause, C. Mayrhofer, and T. Weigand, *G_4 flux, chiral matter and singularity resolution in F-theory compactifications*, *Nucl.Phys.* **B858** (2012) 1–47, [1109.3454]. 53 pages, 2 figures. 3.2.3

[77] J. Marsano and S. Schafer-Nameki, *Yukawas, G-flux, and Spectral Covers from Resolved Calabi-Yau's*, *JHEP* **1111** (2011) 098, [1108.1794]. 3.2.3, 3.2.3

[78] A. P. Braun and T. Watari, *On Singular Fibres in F-Theory*, *JHEP* **1307** (2013) 031, [1301.5814]. 3.2.3

[79] R. Miranda, *The basic theory of elliptic surfaces.* Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989. 3.2.3

[80] M. Kuntzler and C. Lawrie, *Smooth: A Mathematica package for studying resolutions of singular fibrations, Version 0.4*, . 3.2.3

[81] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, vol. 151 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1994. 3.2.3

[82] C. Mayrhofer, D. R. Morrison, O. Till, and T. Weigand, *Mordell-Weil Torsion and the Global Structure of Gauge Groups in F-theory*, *JHEP* **1410** (2014) 16, [1405.3656]. 4

[83] P. S. Aspinwall and D. R. Morrison, *Nonsimply connected gauge groups and rational points on elliptic curves*, *JHEP* **9807** (1998) 012, [hep-th/9805206]. 4

[84] D. S. Park, *Anomaly Equations and Intersection Theory*, *JHEP* **1201** (2012) 093, [1111.2351]. 29 pages + appendices, 8 figures/ v2: minor corrections, references added. 3.2.3

[85] T. Shioda, *Mordell-Weil lattices and Galois representation. I*, *Proc. Japan Acad. Ser. A Math. Sci.* **65** (1989), no. 7 268–271. 3.2.3

[86] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B475** (1996) 94–114, [hep-th/9603142]. 4

[87] C. Saemann and M. Wolf, *Non-Abelian Tensor Multiplet Equations from Twistor Space*, *Commun. Math. Phys.* **328** (2014) 527–544, [1205.3108]. 4

[88] C. Säemann and M. Wolf, *Six-Dimensional Superconformal Field Theories from Principal 3-Bundles over Twistor Space*, *Lett. Math. Phys.* **104** (2014) 1147–1188, [1305.4870]. 4

[89] B. Jurčo, C. Sämann, and M. Wolf, *Semistrict Higher Gauge Theory*, *JHEP* **04** (2015) 087, [1403.7185]. 4

[90] B. Jurco, C. Saemann, and M. Wolf, *Higher Groupoid Bundles, Higher Spaces, and Self-Dual Tensor Field Equations*, *Fortsch. Phys.* **64** (2016) 674–717, [`1604.01639`]. 4

[91] S. Palmer and C. Saemann, *M-brane Models from Non-Abelian Gerbes*, *JHEP* **07** (2012) 010, [`1203.5757`]. 4, 4.6

[92] S. Palmer and C. Sämann, *The ABJM Model is a Higher Gauge Theory*, *Int. J. Geom. Meth. Mod. Phys.* **11** (2014), no. 08 1450075, [`1311.1997`]. 4

[93] E. Cremmer, B. Julia, and J. Scherk, *Supergravity in theory in 11 dimensions*, *Physics Letters B* **76** (1978), no. 4 409 – 412. 4.1

[94] D. S. Berman, *M-theory branes and their interactions*, *Phys. Rept.* **456** (2008) 89–126, [`0710.1707`]. 4.1, 4.1

[95] I. Campbell and P. West, *N = 2, d = 10 non-chiral supergravity and its spontaneous compactification*, *Nuclear Physics B* **243** (1984), no. 1 112 – 124. 4.1

[96] F. Giani and M. Pernici, $n = 2$ *supergravity in ten dimensions*, *Phys. Rev. D* **30** (Jul, 1984) 325–333. 4.1

[97] M. Huq and M. A. Namazie, *Kaluza-klein supergravity in ten dimensions*, *Classical and Quantum Gravity* **2** (1985), no. 3 293. 4.1

[98] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, *Phys. Lett.* **B350** (1995) 184–187, [`hep-th/9501068`]. 4.1, 4.1, 4.1

[99] C. Johnson, *D-Branes*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003. 4.1

[100] K. Becker, M. Becker, and J. H. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 2006. 4.1

[101] S. Mukhi and C. Papageorgakis, *M2 to D2*, *JHEP* **05** (2008) 085, [`0803.3218`]. 4.1

[102] P.-M. Ho, Y. Imamura, and Y. Matsuo, *M2 to D2 revisited*, *JHEP* **07** (2008) 003, [`0805.1202`]. 4.1

[103] M. Aali-Javanangrouh and A. Rezaei-Aghdam, *M2 to D2 and vice versa by 3-Lie and Lie bialgebra*, *Eur. Phys. J.* **C76** (2016), no. 11 632, [`1604.05183`]. 4.1

[104] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, *M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills*, *JHEP* **01** (2011) 083, [1012.2882]. 4.1, 4.7, 4.7, 5

[105] M. R. Douglas, *On D=5 super Yang-Mills theory and (2,0) theory*, *JHEP* **02** (2011) 011, [1012.2880]. 4.1, 4.7, 5

[106] E. Bergshoeff and M. De Roo, *D-branes and T duality*, *Phys. Lett.* **B380** (1996) 265–272, [hep-th/9603123]. 4.1

[107] I. R. Klebanov and A. A. Tseytlin, *Entropy of near extremal black p-branes*, *Nucl. Phys.* **B475** (1996) 164–178, [hep-th/9604089]. 4.2

[108] A. Basu and J. A. Harvey, *The M2-M5 brane system and a generalized Nahm's equation*, *Nucl. Phys.* **B713** (2005) 136–150, [hep-th/0412310]. 4.3, 4.3, 4.3

[109] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys.* **B503** (1997) 220–238, [hep-th/9608163]. 4.3, 4.3

[110] N. R. Constable, R. C. Myers, and O. Tafjord, *The Noncommutative bion core*, *Phys. Rev.* **D61** (2000) 106009, [hep-th/9911136]. 4.3

[111] N. R. Constable, R. C. Myers, and O. Tafjord, *Fuzzy funnels: NonAbelian brane intersections*, in *Strings 2001: International Conference Mumbai, India, January 5-10, 2001*, 2001. hep-th/0105035. 4.3

[112] G. Papadopoulos, *M2-branes, 3-Lie Algebras and Plucker relations*, *JHEP* **05** (2008) 054, [0804.2662]. 4.4

[113] J. P. Gauntlett and J. B. Gutowski, *Constraining Maximally Supersymmetric Membrane Actions*, *JHEP* **06** (2008) 053, [0804.3078]. 4.4

[114] N. Lambert and D. Tong, *Membranes on an Orbifold*, *Phys. Rev. Lett.* **101** (2008) 041602, [0804.1114]. 4.4

[115] N. Lambert and C. Papageorgakis, *Relating U(N)xU(N) to SU(N)xSU(N) Chern-Simons Membrane theories*, *JHEP* **04** (2010) 104, [1001.4779]. 4.4

[116] J. Distler, S. Mukhi, C. Papageorgakis, and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038, [0804.1256]. 4.4

[117] W. Nahm, V. Rittenberg, and M. Scheunert, *The classification of graded lie algebras*, *Physics Letters B* **61** (1976), no. 4 383 – 384. 4.5

[118] J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, *Atomic Classification of 6D SCFTs*, *Fortsch. Phys.* **63** (2015) 468–530, [1502.05405]. 4.5

[119] E. Witten, *Some comments on string dynamics*, in *Future perspectives in string theory. Proceedings, Conference, Strings'95, Los Angeles, USA, March 13-18, 1995*, 1995. hep-th/9507121. 4.5

[120] E. Witten, *Geometric Langlands From Six Dimensions*, 0905.2720. 4.5

[121] P. Pasti, D. P. Sorokin, and M. Tonin, *Note on manifest Lorentz and general coordinate invariance in duality symmetric models*, *Phys. Lett.* **B352** (1995) 59–63, [hep-th/9503182]. 4.5

[122] P. Pasti, D. P. Sorokin, and M. Tonin, *Duality symmetric actions with manifest space-time symmetries*, *Phys. Rev.* **D52** (1995) R4277–R4281, [hep-th/9506109]. 4.5

[123] P. Pasti, D. P. Sorokin, and M. Tonin, *On Lorentz invariant actions for chiral p forms*, *Phys. Rev.* **D55** (1997) 6292–6298, [hep-th/9611100]. 4.5

[124] P. Pasti, D. P. Sorokin, and M. Tonin, *Covariant action for a $D = 11$ five-brane with the chiral field*, *Phys. Lett.* **B398** (1997) 41–46, [hep-th/9701037]. 4.5

[125] S.-L. Ko, D. Sorokin, and P. Vanichchapongjaroen, *The M5-brane action revisited*, *JHEP* **11** (2013) 072, [1308.2231]. 4.5

[126] I. Bandos, H. Samtleben, and D. Sorokin, *Duality-symmetric actions for non-Abelian tensor fields*, *Phys. Rev.* **D88** (2013), no. 2 025024, [1305.1304]. 4.5

[127] C. Papageorgakis and C. Saemann, *The 3-Lie Algebra (2,0) Tensor Multiplet and Equations of Motion on Loop Space*, *JHEP* **05** (2011) 099, [1103.6192]. 4.6

[128] B. Ezhuthachan, S. Mukhi, and C. Papageorgakis, *D2 to D2*, *JHEP* **07** (2008) 041, [0806.1639]. 4.6

[129] N. Lambert and P. Richmond, *(2,0) Supersymmetry and the Light-Cone Description of M5-branes*, *JHEP* **02** (2012) 013, [1109.6454]. 4.6, 5, 5.3

[130] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg, and E. Silverstein, *Matrix description of interacting theories in six-dimensions*, *Adv. Theor. Math. Phys.* **1** (1998) 148–157, [hep-th/9707079]. 4.6, 5

[131] O. Aharony, M. Berkooz, and N. Seiberg, *Light cone description of (2,0) superconformal theories in six-dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119–153, [hep-th/9712117]. 4.6

[132] C. Montonen and D. Olive, *Magnetic monopoles as gauge particles?*, *Physics Letters B* **72** (1977), no. 1 117 – 120. 4.7

[133] D. Gaiotto, *N=2 dualities*, *JHEP* **08** (2012) 034, [0904.2715]. 4.7, 4.7, 6.2.2, 6.2.3

[134] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167–197, [0906.3219]. 4.7, 6

[135] T. Dimofte, D. Gaiotto, and S. Gukov, *Gauge Theories Labelled by Three-Manifolds*, *Commun. Math. Phys.* **325** (2014) 367–419, [1108.4389]. 4.7, 6

[136] H.-J. Chung, T. Dimofte, S. Gukov, and P. Sułkowski, *3d-3d Correspondence Revisited*, *JHEP* **04** (2016) 140, [1405.3663]. 4.7

[137] T. Dimofte, S. Gukov, and L. Hollands, *Vortex Counting and Lagrangian 3-manifolds*, *Lett. Math. Phys.* **98** (2011) 225–287, [1006.0977]. 4.7

[138] A. Gadde, S. Gukov, and P. Putrov, *Fivebranes and 4-manifolds*, 1306.4320. 4.7, 6

[139] C. Vafa and E. Witten, *A Strong coupling test of S duality*, *Nucl. Phys.* **B431** (1994) 3–77, [hep-th/9408074]. 4.7

[140] D. Gaiotto, G. W. Moore, and Y. Tachikawa, *On 6d N=(2,0) theory compactified on a Riemann surface with finite area*, *Prog. Theor. Exp. Phys.* **2013** (2013) 013B03, [1110.2657]. 4.7, 6.2.2, 6.2.2, 6.2.4

[141] W. Taylor, *D-brane field theory on compact spaces*, *Phys. Lett.* **B394** (1997) 283–287, [hep-th/9611042]. 5

[142] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D77** (2008) 065008, [0711.0955]. 5, 5.3

[143] J. Bagger, N. Lambert, S. Mukhi, and C. Papageorgakis, *Multiple Membranes in M-theory*, *Phys. Rept.* **527** (2013) 1–100, [1203.3546]. 5

[144] C. M. Hull and N. Lambert, *Emergent Time and the M5-Brane*, *JHEP* **06** (2014) 016, [1403.4532]. 5, 5.2, 5.3

[145] I. Jeon, N. Lambert, and P. Richmond, *Periodic Arrays of M2-Branes*, *JHEP* **11** (2012) 100, [`1206.6699`]. 5

[146] P.-M. Ho and Y. Matsuo, *M5 from M2*, *JHEP* **06** (2008) 105, [`0804.3629`]. 5

[147] P.-M. Ho, Y. Imamura, Y. Matsuo, and S. Shiba, *M5-brane in three-form flux and multiple M2-branes*, *JHEP* **08** (2008) 014, [`0805.2898`]. 5, 5.3

[148] I. A. Bandos and P. K. Townsend, *Light-cone M5 and multiple M2-branes*, *Class. Quant. Grav.* **25** (2008) 245003, [`0806.4777`]. 5

[149] A. Gustafavsson *Private communication.* 1

[150] U. Gran, *GAMMA: A Mathematica package for performing gamma matrix algebra and Fierz transformations in arbitrary dimensions*, `hep-th/0105086`. 5.1.4, 5.1.5

[151] J. M. Maldacena, A. Strominger, and E. Witten, *Black hole entropy in M theory*, *JHEP* **12** (1997) 002, [`hep-th/9711053`]. 5.2

[152] C. M. Hull and R. R. Khuri, *Branes, times and dualities*, *Nucl. Phys.* **B536** (1998) 219–244, [`hep-th/9808069`]. 5.3

[153] N. J. Hitchin, *On the construction of monopoles*, *Comm. Math. Phys.* **89** (1983), no. 2 145–190. 6, 6.3.1

[154] P. Howe and E. Sezgin, *D = 11, p = 5*, *Physics Letters B* **394** (1997), no. 1 62 – 66. 6.1.2

[155] P. S. Howe, E. Sezgin, and P. C. West, *Covariant field equations of the M theory five-brane*, *Phys. Lett.* **B399** (1997) 49–59, [`hep-th/9702008`]. 6.1.2

[156] C. Cordova and D. L. Jafferis, *Complex Chern-Simons from M5-branes on the Squashed Three-Sphere*, `1305.2891`. 6.1.2

[157] C. Cordova and D. L. Jafferis, *Five-Dimensional Maximally Supersymmetric Yang-Mills in Supergravity Backgrounds*, `1305.2886`. 6.1.2, 6.1.2, 6.1.3, 6.2, 6.2.1, B.3, B.3

[158] T. Kugo and K. Ohashi, *Supergravity tensor calculus in 5-D from 6-D*, *Prog. Theor. Phys.* **104** (2000) 835–865, [`hep-ph/0006231`]. 6.1.2, 6.1.2, 6.2, B.3, B.3

[159] T. Kugo and K. Ohashi, *Off-shell D = 5 supergravity coupled to matter Yang-Mills system*, *Prog. Theor. Phys.* **105** (2001) 323–353, [`hep-ph/0010288`]. 6.1.2

[160] E. Bergshoeff, E. Sezgin, and A. Van Proeyen, *Superconformal Tensor Calculus and Matter Couplings in Six-dimensions*, *Nucl. Phys.* **B264** (1986) 653. [Erratum: Nucl. Phys.B598,667(2001)]. 6.1.2, 6.1.2

[161] E. Bergshoeff, E. Sezgin, and A. Van Proeyen, *(2,0) tensor multiplets and conformal supergravity in $D = 6$*, *Class.Quant.Grav.* **16** (1999) 3193–3206, [hep-th/9904085]. 6.1.2, 6.1.2, 3, 6.1.3, 2

[162] F. Riccioni, *Tensor multiplets in six-dimensional (2,0) supergravity*, *Phys. Lett.* **B422** (1998) 126–134, [hep-th/9712176]. 6.1.2

[163] H. Linander and F. Ohlsson, *(2,0) theory on circle fibrations*, *JHEP* **01** (2012) 159, [1111.6045]. 6.2

[164] E. Witten, *Fivebranes and Knots*, 1101.3216. 6.2.2, 6.2.3

[165] S. K. Donaldson, *Nahm's equations and the classification of monopoles*, *Comm. Math. Phys.* **96** (1984), no. 3 387–407. 6.3.1

[166] M. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopoles.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988. 6.3.1, 6.3.1, 6.3.1, 6.3.1, 6.3.1

[167] E. B. Bogomolny, *The stability of classical solutions*, *Sov. J. Nucl. Phys.* **24** (1976), no. 4 449–454. 6.3.1

[168] D. Gaiotto and E. Witten, *Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory*, 0804.2902. 6.3.1, 6.3.3

[169] R. Bielawski, *Hyper-Kähler structures and group actions*, *J. London Math. Soc. (2)* **55** (1997), no. 2 400–414. 6.3.1

[170] R. Bielawski, *Lie groups, Nahm's equations and hyperkaehler manifolds*, *ArXiv Mathematics e-prints* (Sept., 2005) [math/0509515]. 6.3.1

[171] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, *Topological reduction of 4-d SYM to 2-d sigma models*, *Nucl. Phys.* **B448** (1995) 166–186, [hep-th/9501096]. 6.3.3

[172] J. A. Harvey and A. Strominger, *String theory and the Donaldson polynomial*, *Commun. Math. Phys.* **151** (1993) 221–232, [hep-th/9108020]. 6.3.3

[173] J. P. Gauntlett, *Low-energy dynamics of N=2 supersymmetric monopoles*, *Nucl. Phys.* **B411** (1994) 443–460, [hep-th/9305068]. 6.3.3, B.5.1

[174] P. Deligne, *Courbes elliptiques: formulaire d'après J. Tate*, in *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pp. 53–73. Lecture Notes in Math., Vol. 476. Springer, Berlin, 1975. A.4

[175] T. Nagell, *Sur les propriétés arithmétiques des cubiques planes du premier genre*, *Acta Math.* **52** (1929), no. 1 93–126. A.4

[176] J. W. S. Cassels, *Lectures on elliptic curves*, vol. 24 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1991. A.4